UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

JUNE 2015

PHYS3011 Quantum Mechanics and Electrodynamics (Paper 2)

PHYS3230 Electromagnetism

Time Allowed – 2 hours
Total number of questions – 4
All questions are of equal value
Answer all questions

Answer Q1 & Q2 in one book, and Q3 & Q4 in a second book

Candidates must supply their own, university approved, calculator.

Candidates may bring one A4 page, handwritten on both sides. This must be returned with your exam books.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.

Candidates may keep this paper.

Question 1 (25 marks)

Consider a homogeneous conducting medium that obeys Ohm's law $\mathbf{J}_f = \sigma \mathbf{E}$ and has no net free charge.

(a) Write down Maxwell's equations for this system.

(b) The equations you just wrote down can be solved by

$$\mathbf{E}(z,t) = \left[\tilde{\mathcal{E}} e^{-\kappa z} e^{i(kz-\omega t)} + \text{c.c.}\right] \hat{\mathbf{x}},$$

where

$$k \equiv \omega \sqrt{\frac{\epsilon\mu}{2}} \left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1\right]^{\frac{1}{2}}, \quad \kappa \equiv \omega \sqrt{\frac{\epsilon\mu}{2}} \left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1\right]^{\frac{1}{2}}.$$

Show, using Maxwell's equations, that the corresponding magnetic field is given by

$$\mathbf{B}(z,t) = \sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}} \left[\tilde{\mathcal{E}} e^{-\kappa z} e^{i(kz-\omega t+\phi)} + \text{c.c.}\right] \hat{\mathbf{y}},$$

where the phase ϕ is

$$\phi \equiv \tan^{-1}(\kappa/k).$$

(c) Show that in a good conductor ($\sigma \gg \epsilon\omega$)the magnetic field lags the electric field by 45°.

(d) Show that the skin depths in a poor and a good conductor are, respectively,

$$d = \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}, \qquad \text{if } \sigma \ll \epsilon\omega,$$

$$d = \frac{\lambda}{2\pi}, \qquad \text{if } \sigma \gg \epsilon\omega,$$

where $d \equiv 1/\kappa$, and λ is the wavelength inside the conductor.

Question 2 (25 marks)

An infinitely long cylindrical tube of radius a moves at constant speed v along its axis. It carries a net charge per unit length λ, uniformly distributed over its surface. Surrounding it, at radius b, is another cylinder, moving with the same velocity but carrying the opposite charge $(-\lambda)$.

(a) Find the energy per unit length stored in the fields.

(b) Find the momentum per unit length in the fields.

(c) Find the energy per unit time transported by the fields across a plane perpendicular to the cylinders.

(d) Verify Poynting's theorem at $a < r < b$.

Question 3 (25 marks)

The electromagnetic field strength tensor is:

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}, \tag{1}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{B}$ is the magnetic field, and $c = 1/\sqrt{\mu_0 \epsilon_0}$ is the speed of light.

(a) How many independent components does this tensor have? (1 mark)

(b) Write down expressions for the electric field and the magnetic field in terms of the components of this tensor. (4 marks)

(c) Calculate the value of the scalar $F^{\mu\nu}F_{\mu\nu}$. Do not forget the Einstein summation convention. (4 marks)

(d) The four-vector J^μ is given by $J^\mu = (c\rho, J_x, J_y, J_z)$, where $\boldsymbol{J}$ represents the current density and ρ the charge density. It is known that:

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu, \tag{2}$$

where $\partial_\mu = (\partial/\partial(ct), \partial_x, \partial_y, \partial_z)$. The above equation contains two of Maxwell's equations: one involves the scalar derivative of the electric field, the other involves the curl of the magnetic field. Derive these two Maxwell's equations from Eq. (2). (8 marks)

(e) It is also known that:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0. \tag{3}$$

Remember that here the indices λ, μ, ν are all different, that is, $\lambda \neq \mu \neq \nu$. Use Eq. (3) to derive the other two Maxwell equations: one involving the scalar derivative of the magnetic field, and one involving the curl of the electric field. (8 marks)

Question 4 (25 marks)

(a) The total energy ε of a particle is the sum of two terms: the energy corresponding to its mass and the kinetic energy. Using this, write down an expression for the kinetic energy T involving only the mass m of the particle, its speed v, and the speed of light c. (3 marks)

(b) Consider a particle with mass m and charge q moving with velocity $\boldsymbol{v}$ in an arbitrary electromagnetic field. The covariant equation of motion for this charge is

$$\frac{dp^\mu}{d\tau} = qF^{\mu\nu}v_\nu, \tag{4}$$

where $F^{\mu\nu}$ is the field strength tensor you have encountered previously, p^μ is the four-momentum, v^μ the four-velocity, and τ denotes proper time. Using the space components of this equation, show that the relativistic form of Newton's equation of motion for this charge in terms of the ordinary time t. (6 marks)

(c) Using your result for part (a) to add zero to the time component of Eq. (4), find an equation expressing the rate of change of the kinetic energy of the particle dT/dt. (6 marks)

(d) Consider a particle moving in a uniform, time-independent electric field $\boldsymbol{E} \parallel \hat{x}$ and no magnetic field. By solving Newton's equation of motion found in (b) determine the trajectory $(x(t), y(t))$ for this relativistic particle. You may assume $x(t = 0) = y(t = 0) = 0$. Simplify your final answer by expressing x as a function of y. (10 marks)