Quantum Mechanics – PHYS3210 (maybe?)

THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

Mid-Session test April 2016

Time Allowed – 1 hour

Total number of questions – 2

ALL questions need to be addressed

Question marks for each question are 50.

This paper may be retained by the candidate.

Students may provide their own UNSW approved calculators.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.

Question 1. Quantum oscillator (Marks 50).

Consider the conventional quantum oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\,\omega^2 x^2}{2} \qquad (1)$$

Introduce the creation and annihilation operators

$$\begin{aligned} \hat{a} &= \frac{1}{\sqrt{2}}\left(\frac{x}{b} + b\frac{d}{dx}\right) \\ \hat{a}^+ &= \frac{1}{\sqrt{2}}\left(\frac{x}{b} - b\frac{d}{dx}\right) \end{aligned} \qquad (2)$$

where $b = \sqrt{\hbar/m\,\omega}$ is the oscillator length (for the following calculations it is convenient to choose units in which $b = 1$).

a. Prove that the Hamiltonian (2) can be rewritten as follows

$$\hat{H} = \hbar\,\omega\,(\hat{a}^+\hat{a} + \tfrac{1}{2}). \qquad (3)$$

b. Prove that the energy levels satisfy

$$E_n = \hbar\omega(n + 1/2). \qquad (4)$$

c. Take the set of the wave functions $\psi_n(x) \equiv |n\rangle$, which correspond to the energy levels E_n. Find explicitly the matrix elements $\langle m|x|n\rangle =?$ and $\langle m|\hat{p}|n\rangle = ?$

Hint. Remember the relation

$$\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \qquad (5)$$

which was discussed in lectures, and take into account that

$$\langle m|\hat{a}^+|n\rangle = \langle m|\hat{a}^+|n\rangle^* = \langle n|\hat{a}|m\rangle\,. \qquad (10)$$

(the first identity here holds because the functions are real-valued, the second is the definition of the Hermition conjugate).

Question 2. Semiclassical approximation (Marks 50).

Consider a negative ion which outer electron has the electron affinity B (to avoid confusion, B is the energy, which is necessary to separate the electron from the atomic particle, i.e. the binding energy).
Assume that the ion is placed in a weak homogeneous electric field E, which satisfies condition

$$eEa \ll B\,, \qquad (11)$$

where a is the typical size of the ion. Remember that the potential energy of the electron in the homogeneous field reads $U_E = -eEx$.

a. Outline in simple physical terms the reason, which makes the ion in the electric field unstable.
b. Present a qualitative sketch for the electron wave function outside the atomic particle. Distinguish two cases, the free ion and the ion placed in the electric field.
c. With exponential accuracy find the rate for the probability of the ion decay (i.e. separation of the electron from the ion).

Hint. To facilitate calculations remember the integral

$$\int_0^{a/b} \sqrt{a - bx}\, dx = \frac{2}{3}\frac{a^{3/2}}{b} \qquad (12)$$