Question 1. Properties of the Schrödinger equation (Marks 40).
Assume that a particle of mass m propagates along the x-axis in the attractive, smooth potential $U(x)$ being thus described by the conventional Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + U(x). \tag{1}$$

Let $\psi_1(x)$ and $\psi_2(x)$ be the wave functions describing two discrete energy levels, which exist in this potential,

$$\hat{H}\,\psi_n(x) = E_n \psi_n(x), \quad n = 1,2, \tag{2}$$

where $E_1, E_2 < 0, \;\; E_1 \neq E_2$.

a. Formulate the boundary conditions for these functions at infinity, outline briefly their physical meaning.
b. Prove that these wave functions can be chosen to be real.
c. Prove that these wave functions are orthogonal, i. e. verify that

$$\langle\psi_1|\psi_2\rangle \equiv \int_{-\infty}^{\infty} \psi_1^*(x)\,\psi_2(x)\,dx = 0\,. \tag{3}$$

Hint: use Eqs.(2) and the fact that the Hamiltonian is Hermitian, which means that

$$\int_{-\infty}^{\infty} \psi_1^*(x)\,\hat{H}\,\psi_2(x)\,dx = \int_{-\infty}^{\infty} (\hat{H}\psi_1(x)\,)^*\,\psi_2(x)\,dx\,. \tag{4}$$

Comment. Since the wave functions are real valued, see Q1 b, the sign of the complex conjugation * is excessive here, but is kept for the sake of conventional notation.

Question 2. Quantum oscillator (Marks 60).
Consider the conventional quantum oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\,\omega^2 x^2}{2} = \hbar\,\omega\,(\hat{a}^+\hat{a} + \tfrac{1}{2}). \tag{5}$$

Here the creation and annihilation operators read

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{x}{b} + b\frac{d}{dx}\right), \qquad \hat{a}^+ = \frac{1}{\sqrt{2}}\left(\frac{x}{b} - b\frac{d}{dx}\right), \tag{6}$$

where $b = \sqrt{\hbar/m\,\omega}$ is the typical for the quantum oscillator length (for the following calculations it can be convenient to choose units in which $b = 1$ and recover its explicit value only at the end of calculations).

Call $\psi_n(x) \equiv |n\rangle$ the wave function that describes the n-th energy level E_n; this means that this wave function satisfies equation

$$\hat{H}|n\rangle = E_n|n\rangle\,. \tag{7}$$

Remember that the creation operator $\hat{a}^{+}$ generates the following chain-relation between wave functions of consequent energy levels

$$\hat{a}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{8}$$

(it was derived in the classroom).

a. Prove that the annihilation operator generates a similar relation

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle. \tag{9}$$

Hint. One possible simple way is to apply to both sides of Eq.(8) the operator $\hat{a}$ and after that use the commutation relation for the creation and annihilation operators.

b. Derive explicit expressions for the matrix elements

$$\langle m|x|n\rangle = \int_{-\infty}^{\infty} \psi_m^*(x)\, x\, \psi_n(x)\, dx \tag{10}$$

and

$$\langle m|\frac{d}{dx}|n\rangle = \int_{-\infty}^{\infty} \psi_m^*(x)\, \frac{d}{dx}\, \psi_n(x)\, dx. \tag{11}$$

(In other words, find simple, explicit formulas, which describe dependence of these matrix elements on intgers m and n.)

Hint: it is convenient firstly to calculate the matrix elements $\langle m|\hat{a}^{+}|n\rangle$ and $\langle m|\hat{a}\,|n\rangle$ using Eqs.(8), (9) and (3), and after that take into account definitions (6).