THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

PHYS3210 – Quantum Mechanics
PHYS3011 – Quantum Mechanics + Electrodynamics, Paper 1

Session 1, 2015

1. Time allowed – 2 hours
2. Total number of questions – 4
3. Total marks available – 100
4. Answer ALL questions. If math presents a difficulty use physical arguments and plain English.
5. **Answer Part A (questions 1, 2) in one booklet and Part B (questions 3, 4) in a separate booklet.**
6. QUESTIONS ARE NOT OF EQUAL VALUE.
   Marks available for each question are shown in the examination paper.
7. University-approved calculators may be used.
8. All answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.
9. This paper may be retained by the candidate.

Useful Formulae

- Spherical harmonics $Y_{lm}(\theta, \phi)$

$$
\begin{aligned}
Y_{00} &= \sqrt{\frac{1}{4\pi}} \\
Y_{10} &= \sqrt{\frac{3}{4\pi}}\cos\theta \\
Y_{1\pm1} &= \mp\sqrt{\frac{3}{8\pi}}\, e^{\pm i\phi}\sin\theta \\
Y_{20} &= \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1) \\
Y_{2\pm1} &= \mp\sqrt{\frac{15}{8\pi}}\, e^{\pm i\phi}\cos\theta\sin\theta \\
Y_{2\pm2} &= \sqrt{\frac{15}{32\pi}}\, e^{\pm 2i\phi}\sin^2\theta
\end{aligned}
$$

- Wavefunction of K-shell electron (1s electron):

$$
\psi(r) = \sqrt{\frac{Z^3}{\pi a_B^3}}\exp(-Zr/a_B), \qquad a_B = \frac{\hbar^2}{m_e e^2}
$$

Part A (answer in a separate booklet)

Question 1 (20 marks)

Quantum oscillator

Consider the conventional 1D quantum oscillator described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 x^2}{2}, \quad \hat{p} = -i\hbar\frac{d}{dx} \tag{1}$$

(a) Verify that the wave function

$$\psi_0(x) = \frac{1}{\pi^{1/4}\sqrt{b}} \exp\left(-\frac{x^2}{2b^2}\right), \quad b = \sqrt{\frac{\hbar}{m\omega}} \tag{2}$$

satisfies the stationary Schrödinger equation. Find the corresponding energy. Show that this wave function describes the ground state.

(b) Derive the wave functions for the two first excited states $\psi_1(x)$ and $\psi_2(x)$. *Hint:* Remember the creation and annihilation operators

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{x}{b} + b\frac{d}{dx}\right) \tag{3}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x}{b} - b\frac{d}{dx}\right) \tag{4}$$

as well as the expression for the ground state wave function Eq. (2).

(c) Consider a slightly modified 3D harmonic oscillator described by the Hamiltonian

$$\hat{H} = \frac{\hat{\boldsymbol{p}}^2}{2m} + \frac{m\omega^2 r^2}{2} + \boldsymbol{V}\cdot\hat{\boldsymbol{p}} - \boldsymbol{F}\cdot\boldsymbol{r}, \tag{5}$$

where $\boldsymbol{r}$ is the radius vector and $\hat{\boldsymbol{p}} = -i\hbar\nabla$ is the corresponding momentum, while $\boldsymbol{V}$, $\boldsymbol{F}$ are two real-valued constant vectors.

i. Suggest a possible physical system that explains this $\hat{H}$ (i.e. suggest meanings for the extra terms).

ii. Find the energy spectrum of $\hat{H}$ (i.e. present all energy levels in terms of the constants given in Eq. (5): $\hbar$, m, ω, $\boldsymbol{V}$, $\boldsymbol{F}$).

Question 2 (30 marks)

Semiclassical approximation and related topics

(a) Consider the Hamiltonian

$$\hat{H} = \frac{\hat{\boldsymbol{p}}^2}{2m} + U(r), \quad U(r) = \begin{cases} -W, & r \le a \\ 0, & r > a \end{cases} \tag{6}$$

which describes propagation of a particle in 3D with the potential energy that represents the potential well of radius a and depth $W > 0$; a, W are given constants.

i. Using the Heisenberg uncertainty principle find the condition on parameters a, W which indicates that a particle of mass m has a bound state in this well. Use the same condition to find the situation when there is no binding.

ii. Suppose that the electron has a bound state in the electrostatic potential well $U(r)$. What happens with a muon in the same potential well: would it show no bound states, a bound state, or a lot of them? Substantiate your assessment.

(b) Consider the α-decay of a heavy nucleus of charge Z. Let the energy of the α-particle be $E = Mv^2/2$, where M is the mass while v is the velocity of the α-particle after the decay, when the α-particle is at a large distance from the (very small) nucleus. The Hamiltonian that describes the propagation of the α-particle outside the nucleus reads

$$\hat{H} = \frac{\hat{p}^2}{2M} + \frac{Z' Z_\alpha e^2}{r} \,. \tag{7}$$

Here $Z' = Z - 2$ is the charge of the residual nucleus, $Z_\alpha = 2$ is the charge of the α-particle, r is the separation of the α-particle from the nuclear residue (the kinetic energy of the residue is neglected in (7) because the nucleus is heavy, which makes the recoil small).

i. Present a sketch of the potential energy from (7); show on this sketch the energy of the α-particle E and the location of the classically forbidden area, which separates a region of very small inner-nuclear distances from the region of sufficiently large, classically allowed distances.

ii. Estimate with exponential accuracy the probability of the decay, which allows the α-particle to negotiate the forbidden region and escape the nucleus. Hint: the necessary probability can be estimated as follows

$$W \propto \psi^2$$

where ψ describes the exponential suppression of the wave function in the classically forbidden interval of separations. This suppression can be found using the semiclassical approximation. The calculations require only the integration, which essentially reads

$$\int_0^{\alpha/\beta} \sqrt{\frac{\alpha}{x} - \beta dx} = \frac{\pi\alpha}{2\sqrt{\beta}}\,, \quad \alpha, \beta > 0\,.$$

It may be helpful to keep in mind that you need to derive a clear, simple formula for W (called Sommerfeld's factor). It turns out that the probability is a function of only one dimensionless parameter, $\frac{Z' Z_\alpha e^2}{\hbar v}$, which means $W = W(\frac{Z' Z_\alpha e^2}{\hbar v})$.

iii. Outline briefly why the probability of the α-decay varies so strongly, by many orders of magnitude for different nuclei.

Part B (answer in a separate booklet)

Question 3 (20 marks)

Field shift in muonic atoms

A muonic atom consists of a μ meson (mass $m_\mu \approx 200 m_e$) bound to a proton in a hydrogenic orbit. The energies of the μ-meson levels are shifted relative to their values for a point nucleus because the nuclear charge is distributed over a region with radius R. For simplicity, the effective Coulomb potential can be approximated as

$$V(r) = \begin{cases} -\frac{e^2}{r}, & r \geq R \\ -\frac{e^2}{R}, & r \leq R \end{cases} \tag{8}$$

That is, the ordinary point-like nuclear potential is "cut off": constant at $r \leq R$.

(a) State qualitatively how the energies of the $1s$, $2s$, and $2p$ muonic levels will be shifted, both in absolute terms and relative to each other, and explain physically any differences in the shifts. Sketch an energy level diagram with both unperturbed and perturbed levels for these states.

(b) Calculate the first order change in energy of the $1s$ state associated with the fact that the nucleus is not point-like. You may assume that $R/a_\mu \ll 1$, where $a_\mu = \frac{\hbar^2}{m_\mu e^2}$ is the "Bohr radius" for the muon.

(c) Estimate the $2s$-$2p$ energy shift under the assumption that $R/a_\mu \ll 1$. Show that measurement of this shift could be used to measure R. How much smaller is the effect in an ordinary electronic atom?

Useful formulae:

$$\psi_{1s} = \frac{2}{a_\mu^{3/2}} e^{-r/a_\mu} Y_{00}$$

$$\psi_{2s} = \frac{1}{\sqrt{8}\, a_\mu^{3/2}} \left(2 - \frac{r}{a_\mu}\right) e^{-r/2a_\mu} Y_{00}$$

$$\psi_{2p} = \frac{1}{\sqrt{24}\, a_\mu^{3/2}} \frac{r}{a_\mu} e^{-r/2a_\mu} Y_{1m}$$

Question 4 (30 marks)

Variational principle vs. perturbation theory

Consider a system in three-dimensions which can be approximated using only two bound states

$$|\psi_0\rangle = R_0(r)\, Y_{00}(\theta, \phi) \tag{9}$$

$$|\psi_{1m}\rangle = R_1(r)\, Y_{1m}(\theta, \phi) \tag{10}$$

which have energies ϵ_0 and ϵ_1 and orbital angular momentum $l = 0$ and $l = 1$, respectively. The upper state, $|\psi_{1m}\rangle$, is triply-degenerate.

A weak electric field is applied in the z-direction, resulting in the perturbation

$$\delta V = E\, z = E\, r \cos\theta \tag{11}$$

(a) Show that the first-order energy shift for $|\psi_0\rangle$ and $|\psi_{1m}\rangle$ is zero. You may make use of the formulas

$$\int_0^\pi \sin\theta \cos^n\theta \, d\theta = \begin{cases} \frac{2}{1+n}, & n = 0, 2, 4, ... \\ 0, & n = 1, 3, ... \end{cases}$$

$$\int_0^\pi \sin^n\theta \cos\theta \, d\theta = 0$$

(b) Write an expression for the matrix element

$$W = \langle\psi_0|\, \delta V \,|\psi_{10}\rangle \tag{12}$$

Leave your answer in terms of integrals over the radial functions R_0 and R_1, but evaluate the angular part.

(c) Write an expression for the second-order perturbation theory correction to the energies of the states $|\psi_0\rangle$ and $|\psi_{10}\rangle$ in terms of ϵ_0 and ϵ_1 and W.

(d) The Hamiltonian of the unperturbed system (for $m = 0$) can be represented as a two-by-two matrix

$$\hat{H}_0 = \begin{pmatrix} \epsilon_0 & 0 \\ 0 & \epsilon_1 \end{pmatrix}$$

while the wave functions can be represented as the vectors

$$|\psi_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\psi_{10}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

so that $\hat{H}_0\,|\psi_i\rangle = \epsilon_i\,|\psi_i\rangle$ for $i = 0, 1$.

The full Hamiltonian in this basis is

$$\hat{H} = \hat{H}_0 + \delta V = \begin{pmatrix} \epsilon_0 & W \\ W & \epsilon_1 \end{pmatrix} \tag{13}$$

since the perturbation δV does not mix states with different m. Find the energies of the system (eigenvalues of the Hamiltonian).

Hint: The eigenvalues λ of a matrix $\mathbf{A}$ can be found by finding the roots of the characteristic equation: $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, which in the case of a two-by-two matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is simply

$$(a - \lambda)(d - \lambda) - bc = 0\,.$$

(e) Show that for small electric fields E, and hence small W, the exact eigenvalues correspond to the results of second-order perturbation theory.

Hint: $\sqrt{1 + x} \approx 1 + x/2$ for small x.

— End of Exam —