THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

JUNE/JULY 2013

PHYS 3210 Quantum Mechanics

PHYS 3011 Quantum Mechanics + Electrodynamics (Paper 2)

PHYS 9583 Advanced Theoretical Physics 2 (Paper 1)

Time Allowed – 2 hours

Total number of questions – 3

ALL questions need to be addressed

Question marks range from 30 to 40.

This paper may be retained by the candidate.

Students may provide their own UNSW approved calculators.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.

Question 1. Semiclassical approximation (Marks 30).

Consider the decay of the heavy nucleus of charge $Z+1$ into the nucleus of charge Z and the proton, which has the mass m. Let the energy of the proton equals $E > 0$, $E = \frac{m v^2}{2}$, where v is the proton velocity at large separations.

a. Write down the potential energy $U(r)$ of the proton in the Coulomb field created by the residual nucleus of charge Z.
b. Remember that in the classical approximation the repulsive potential energy $U(r)>0$ prevents the proton from being located in the proximity of the nucleus. Find the stopping point r_0, which separates the classically allowed region of large separations, $r > r_0$, from the forbidden region $r_N < r < r_0$. Here r_N is the radius of the residual nucleus, which usually, for not extremely high E, is much smaller than r_0, and hence can be approximated by zero, $r_N \approx 0$. Presume in these estimations that the orbital momentum of the proton is zero. Present the sketch for the potential energy, show in this sketch the energy level E and the point r_0.
c. Write down the Hamiltonian, which governs propagation of the proton outside the nucleus with zero angular momentum.
d. With exponential accuracy estimate how the probability of the decay depends on the energy E. With this purpose estimate the probability for the quantum tunneling, which allows the proton to negotiate the classically forbidden region, $0 < r < r_0$.

Hint. The necessary probability can be estimated as follows

$$W \propto \psi^2, \qquad (1.1)$$

where ψ describes the exponential suppression of the wave function in the classically forbidden interval of separations $0 \leq r \leq r_0$. This suppression can be found using the semiclassical approximation. The calculations require the following integral

$$\int_0^{r_0} \sqrt{\frac{1}{r} - \frac{1}{r_0}}\, dr = \frac{\pi}{2}\sqrt{r_0}\,. \qquad (1.2)$$

It may be helpful to keep in mind that you need to derive a clear, simple formula for W, called Rutherford's factor, which is a function of only one dimensionless parameter, $W = W(\frac{Ze^2}{\hbar v})$.

Question 2. Coulomb problem (Marks 30).

a. Write down the Schrödinger equation for the radial wave function $P_l(r)$, which describes the radial motion of the electron with the orbital momentum l in the Hydrogen atom. Remember that the wave function in this case can be presented as follows

$$\psi(\boldsymbol{r}) = \frac{1}{r} P_l(r)\, Y_{l,m}(\theta,\phi)\,. \qquad (2.1)$$

Keep in mind also that in the spherical coordinates the Laplacian reads

$$\Delta = \Delta_r + \frac{\Lambda}{r^2}\,, \qquad (2.2)$$

while the spherical harmonic $Y_{l,m}(\theta,\phi)$ is the eigenfunction of the angular part of the Laplacian Λ

$$-\Lambda\, Y_{l,m}(\theta,\phi) = l(l+1) Y_{l,m}(\theta,\phi)\,. \qquad (2.3)$$

It is also helpful to remember that

$$\Delta_r\left(\frac{P_l(r)}{r}\right) = \frac{1}{r}\, P_l''(r)\,. \qquad (2.4)$$

b. Consider the following p-wave function, i. e. a radial function that describes a state with $l = 1$,

$$P_{2p}(r) = \frac{1}{\sqrt{24\, a_B}}\, (r/a_B)^2 \exp\left(-\frac{r}{2\, a_B}\right), \qquad (2.5)$$

where $a_B = \hbar/(me^2)$ is the Bohr radius. Prove that $P_{2p}(r)$ satisfies the radial Schrödinger equation and find the corresponding eigenvalue, i. e. the energy E_{2p}.
Hint: here and below it is convenient to fulfill calculations in atomic units, in which $\hbar = m_e = |e| = a_B = 1$.

c. Prove that the wave function (2.5) describes the lowest energy state for the p-wave.

d. Verify validity of the normalization condition

$$\int_0^\infty P_{2p}^2(r)\, dr = 1\,. \qquad (2.6)$$

Hint: the necessary integral reads

$$\int_0^\infty e^{-\gamma x} x^n\, dx = \frac{n!}{\gamma^{n+1}}\,. \qquad (2.7)$$

Question 3. Perturbation theory (Marks 40).

a. Consider the 2p state of the Hydrogen atom. Assume that there exists the small correction to the Coulomb field

$$\delta\, U(r) = \lambda \exp(-\mu\, r)\,. \qquad (3.1)$$

where λ, μ are two parameter, λ is small and $\mu > 0$.
Using the perturbation theory find the first order correction δE_{2p} to the energy of the 2p state.
Hint. Remember that the related wave function is known, see (2.5), and keep in mind that the necessary in calculations integral is given in (2.7).

b. Consider the perturbation theory in general case. Assume that there exist two close energy levels, $E_1 < E_2$, which separation is much smaller then energy separations with other energy levels in the system

$$E_2 - E_1 \ll |E_2 - E_n|,\ |E_1 - E_n|,\ n = 3, 4 \ldots \qquad (3.2)$$

Assume further that there exists some perturbation δU, and that the first order corrections to E_1 and E_2 are absent, $\langle 1|\delta U|1\rangle = \langle 2|\delta U|2\rangle = 0$.
Prove that the second order corrections to these states satisfy

$$\delta^{(2)} E_2 \approx -\delta^{(2)} E_1,\quad \delta^{(2)} E_1 < 0,\quad \delta^{(2)} E_2 > 0\,, \qquad (3.3)$$

which make the energy separation between the two close energy levels bigger (the effect is often referred to as repulsion of close energy levels).

c. Prove the following statement. (Sometimes it is called the Hellmann–Feynman theorem, but do not be scared out by names, it is a simple statement.)
Suppose there is the Hamiltonian $\hat{H} = \hat{H}(\lambda)$, which depends on the parameter λ (an example provides Eq.(3.1), see also (3.7) below). Correspondingly, all energy levels in the system as well as all wave functions describing these levels depend on this parameter, $E_n = E_n(\lambda)$, $\psi_n = \psi_n(\lambda)$. Prove that the derivative of the energy satisfies

$$\frac{dE_n(\lambda)}{d\lambda} = \langle \psi_n(\lambda)|\frac{\partial \hat{H}(\lambda)}{\partial \lambda}|\psi_n(\lambda)\rangle. \qquad (3.4)$$

Hint. Use the fact that

$$E_n(\lambda) = \langle \psi_n(\lambda)\left|\hat{H}(\lambda)\right|\psi_n(\lambda)\rangle, \qquad (3.5)$$

as well as an obvious identity

$$\hat{H}(\lambda + \delta\lambda) \approx \hat{H}(\lambda) + \delta\lambda\, \frac{\partial \hat{H}}{\partial \lambda}(\lambda). \qquad (3.6)$$

Apply after that the first order perturbation theory for $\delta\lambda\, \frac{\partial \hat{H}}{\partial \lambda}$.

d. Verify that (3.4) is valid for the example considered in Question 3a, when

$$\hat{H}(\lambda) = \frac{\hat{p}^2}{2m} + \lambda e^{-\mu r}. \qquad (3.7)$$