THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

MID-TERM EXAMINATION

APRIL 2016

PHYS3021 Statistical and Solid State Physics - PAPER 1

PHYS3020 Statistical Physics

Time Allowed – 50 minutes

Total number of questions - 2

Answer ALL questions

This exam is worth 7.5% of the final grade for PHYS3021 students

This exam is worth 15% of the final grade for PHYS3020 students

Candidates must supply their own university approved calculator.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.

This paper may be retained by the candidate

FORMULA SHEET

Boltzmann Entropy $S = k \ln W$

Statistics and Distributions

Boltzmann $W_B = N! \prod_{j=1}^{n} \frac{g_j^{N_j}}{N_j!}$ $\frac{N_j}{g_j} = \frac{N}{Z} e^{-\varepsilon_j/kT}$ $Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT}$

Maxwell-Boltzmann $W_{MB} = \prod_{j=1}^{n} \frac{g_j^{N_j}}{N_j!}$ $\frac{N_j}{g_j} = \frac{N}{Z} e^{-\varepsilon_j/kT}$ $Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT}$

Fermi-Dirac $W_{FD} = \prod_{j=1}^{n} \frac{g_j!}{N_j!(g_j - N_j)!}$ $\frac{N_j}{g_j} = \frac{1}{e^{(\varepsilon_j - \mu)/kT} + 1}$

Bose-Einstein $W_{BE} = \prod_{j=1}^{n} \frac{(N_j + g_j - 1)!}{N_j!(g_j - 1)!}$ $\frac{N_j}{g_j} = \frac{1}{e^{(\varepsilon_j - \mu)/kT} - 1}$

Microcanonical $f_{mc}(q,p) = \frac{\delta(H(q,p) - E)}{\int dqdp\,\delta(H(q,p) - E)}$

Canonical $f_C(q,p) = \frac{\exp(-\beta H(q,p))}{Z(N,V,T)}$ $Z(N,V,T) = \int dqdp \exp(-\beta H(q,p))$

Grand-canonical $f_G(q,p) = \frac{\exp(\beta(\mu N - H))}{\Xi(\mu,V,T)}$ $\Xi(\mu,V,T) = \sum_{N=0}^{\infty} z^N Z(N,V,T)$

Thermodynamic Potentials

Internal energy	U	$dU = TdS - PdV$
Enthalpy	$H = U + PV$	$dH = TdS + VdP$
Helmholtz function	$F = U - TS$	$dF = -SdT - PdV$
Gibbs function	$G = U - TS + PV$	$dG = -SdT + VdP$

Statistical Mechanics Canonical Ensemble

Internal energy $U = Nk_B T^2 \left(\frac{\partial \ln Z}{\partial T} \right)_V$ Pressure $P = Nk_B T \left(\frac{\partial \ln Z}{\partial V} \right)_T$

Mathematical identities

$\ln N! \approx N \ln N - N$

$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$

$\frac{x}{5} = 1 - e^{-x} \quad \Rightarrow \quad x = 4.96$

$1 + y + y^2 + \ldots = \frac{1}{1-y}$

$\int_{-\infty}^{\infty} dx e^{-x^2/\alpha} = \sqrt{\pi\alpha}$

$\int_0^\infty d\varepsilon \varepsilon^{1/2} e^{-\varepsilon/\alpha} = \frac{\alpha}{2}\sqrt{\pi\alpha}$

$\sinh(x) = \frac{e^x - e^{-x}}{2}$

$\frac{d}{dx}\sinh(x) = \cosh(x)$

$\mathrm{csch}(x) = \frac{1}{\sinh(x)}$

$\cosh(x) = \frac{e^x + e^{-x}}{2}$

$\frac{d}{dx}\cosh(x) = \sinh(x)$

$\mathrm{sech}(x) = \frac{1}{\cosh(x)}$

$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$

$\frac{d}{dx}\tanh(x) = \mathrm{sech}^2(x)$

$\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

$\coth(x) = \frac{\cosh(x)}{\sinh(x)}$

$\frac{d}{dx}\coth(x) = -\mathrm{csch}^2(x)$

$\int_{-\infty}^{\infty} \frac{e^y dy}{\left(e^y + 1\right)^2} = 1$

$\int_{-\infty}^{\infty} \frac{y^2 e^y dy}{\left(e^y + 1\right)^2} = \frac{\pi^2}{3}$

QUESTION 1 (15 marks)

(a) The energy of a system is given by $E = \alpha x^2$. The system is in contact with a large reservoir which has a constant temperature *T*. Show that the mean energy of the system is $k_B T/2$.

(8 marks)

(b) If the system has *n* such quadratic degrees of freedom, what is its mean thermal energy? What is the mean thermal energy of a diatomic gas when only the translational and rotational degrees of freedom are taken into account? What is the mean thermal energy if the vibrational degrees of freedom are also taken into account?

(3 marks)

(c) The relationship you proved above is called the equipartition of energy. What are the two key assumptions leading to this relationship, and in what range of temperatures is it valid?

(4 marks)

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QUESTION 2 (15 marks)

Consider an ideal gas of particles of mass m. The Maxwell-Boltzmann distribution is applicable here. Each level has occupancy N_j and degeneracy g_j.

(i) The thermodynamic probability W for the Maxwell-Boltzmann distribution takes the form

$$W = \prod_{j=0}^{n} \frac{g_j^{N_j}}{N_j!}$$

Use this expression and the statistical definition of entropy to determine the entropy S as a function of the particle number N, partition function Z, and internal energy U. (5 marks)

(ii) The partition function Z for an ideal gas is:

$$Z = V\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}$$

where V is the volume. Using the known expressions for the pressure p and internal energy U in terms of Z, evaluate p and U explicitly. (5 marks)

(iii) Use your results from (i) and (ii) to work out S, and show that the the entropy per mol, given by $s = S/n_m$ (n_m is the number of moles), satisfies

$$s = s_0 + R\ln V + c_v \ln T$$

Make sure you identify the constant s_0 and the molar heat capacity c_v. (5 marks)