THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

JUNE 2013

PHYS3021 Statistical and Solid State Physics - PAPER 1

PHYS3020 Statistical Physics

Time Allowed – 2 hours

Total number of questions - 5

Answer ALL questions

All questions ARE of equal value

This exam is worth 35% of the final grade for PHYS3021 students

This exam is worth 70% of the final grade for PHYS3020 students

Candidates must supply their own university approved calculator.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.

FORMULA SHEET

Boltzmann Entropy $S = k \ln W$

Statistics and Distributions

Boltzmann	$W_B = N! \prod_{j=1}^{n} \frac{g_j^{N_j}}{N_j!}$	$\frac{N_j}{g_j} = \frac{N}{Z} e^{-\varepsilon_j/kT}$	$Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT}$
Maxwell-Boltzmann	$W_{MB} = \prod_{j=1}^{n} \frac{g_j^{N_j}}{N_j!}$	$\frac{N_j}{g_j} = \frac{N}{Z} e^{-\varepsilon_j/kT}$	$Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT}$
Fermi-Dirac	$W_{FD} = \prod_{j=1}^{n} \frac{g_j!}{N_j!(g_j - N_j)!}$		$\frac{N_j}{g_j} = \frac{1}{e^{(\varepsilon_j - \mu)/kT} + 1}$
Bose-Einstein	$W_{BE} = \prod_{j=1}^{n} \frac{(N_j + g_j - 1)!}{N_j!(g_j - 1)!}$		$\frac{N_j}{g_j} = \frac{1}{e^{(\varepsilon_j - \mu)/kT} - 1}$
Microcanonical	$f_{mc}(q,p) = \frac{\delta(H(q,p) - E)}{\int dq dp \delta(H(q,p) - E)}$		
Canonical	$f_C(q,p) = \frac{\exp(-\beta H(q,p))}{Z(N,V,T)}$	$Z(N,V,T) = \int dq dp \exp(-\beta H(q,p))$	
Grand-canonical	$f_G(q,p) = \frac{\exp(\beta(\mu N - H))}{\Xi(\mu,V,T)}$	$\Xi(\mu,V,T) = \sum_{N=0}^{\infty} z^N Z(N,V,T)$	

Thermodynamic Potentials

Internal energy	U	$dU = TdS - PdV$
Enthalpy	$H = U + PV$	$dH = TdS + VdP$
Helmholtz function	$F = U - TS$	$dF = -SdT - PdV$
Gibbs function	$G = U - TS + PV$	$dG = -SdT + VdP$

Statistical Mechanics Canonical Ensemble

Internal energy $U = kT^2 \left(\frac{\partial \ln Z}{\partial T} \right)_V$ Pressure $P = kT \left(\frac{\partial \ln Z}{\partial V} \right)_T$

Mathematical identities

$\ln N! \approx N \ln N - N$

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$$

$$\frac{x}{5} = 1 - e^{-x} \quad \Rightarrow \quad x = 4.96$$

$$1 + y + y^2 + \ldots = \frac{1}{1-y}$$

$$\int_{-\infty}^{\infty} dx e^{-x^2/\alpha} = \sqrt{\pi\alpha}$$

$$\int_0^\infty d\varepsilon\, \varepsilon^{1/2} e^{-\varepsilon/\alpha} = \frac{\alpha}{2}\sqrt{\pi\alpha}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\frac{d}{dx}\sinh(x) = \cosh(x)$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\frac{d}{dx}\cosh(x) = \sinh(x)$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$$

$$\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)}$$

$$\frac{d}{dx}\coth(x) = -\operatorname{csch}^2(x)$$

$$\int_{-\infty}^{\infty} \frac{e^y dy}{(e^y + 1)^2} = 1$$

$$\int_{-\infty}^{\infty} \frac{y^2 e^y dy}{(e^y + 1)^2} = \frac{\pi^2}{3}$$

QUESTION 1 (20 marks)

Part I

(a) Consider a system in which the allowed energy levels $0, \varepsilon, 2\varepsilon, 3\varepsilon, 4\varepsilon, \ldots..$ are nondegenerate ($g_j = 1$). If the system has 3 distinguishable particles and a total energy of $U = 6\varepsilon$, tabulate the possible distributions of the particles among the energy levels and calculate the thermodynamic weight of each macrostate.

(b) Explain how this tabulation changes if the particles are indistinguishable bosons.

(c) Construct the same table of macrostates and thermodynamic weights for indistinguishable fermions.

(d) Calculate the average occupancy of each energy level for the fermion system.

(e) Estimate the Fermi energy (or chemical potential) $\varepsilon_F = \mu(0)$ for the system of 3 fermions.

Part II.

(a) For an open system the change in internal energy is given by $dU = TdS - PdV + \mu dN$. Show that the Maxwell Boltzmann distribution can be written in a form containing the chemical potential μ. Thus show that

$$f_j = \frac{N_j}{g_j} = \frac{N}{Z} e^{-\varepsilon_j/kT} = \frac{1}{e^{(\varepsilon_j - \mu)/kT}}.$$

QUESTION 2 (20 marks)

(a) Show that the thermodynamic probability

$$W = \prod_{j=1}^{n} \frac{g_j(g_j - a)(g_j - 2a)...(g_j - (N_j - 1)a)}{N_j!}$$

Reduces to Maxwell-Boltzmann statistics when $a = 0$, to Fermi-Dirac statistics when $a = 1$, and to Bose-Einstein statistics when $a = -1$.

(b) If the density of states $g(\varepsilon)d\varepsilon = \left(4\sqrt{2}\pi V/h^3\right)m^{3/2}\varepsilon^{1/2}d\varepsilon$, show that the sum can be approximated by an integral to give the partition function

$$Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT} = \left(\frac{2\pi mkT}{h^2}\right)^{3/2} V.$$

Calculate the internal energy and pressure for this system using

$$U = NkT^2\left(\frac{\partial \ln Z}{\partial T}\right)_V \quad \text{and} \quad P = NkT\left(\frac{\partial \ln Z}{\partial V}\right)_T.$$

(c) If the Helmholtz function $F = -NkT(\ln(Z/N) + 1)$, show that the entropy of the system is

$$S = Nk\left(\frac{3}{2}\ln T - \ln\left(\frac{N}{V}\right)\right) + S_0$$

and find the value of the constant S_0.

d) If the partition function of a system that obeys Maxwell-Boltzmann statistics is given by $Z = aV^{\alpha}T^{\beta}$, where a is a constant, find the relationship between the pressure P and internal energy per unit volume U/V.

QUESTION 3 (20 marks)

Part I

(a) For a classical system of N particles in three dimensions where the energy varies continuously show that the kinetic contribution to the canonical partition function is given by

$$Z_K = \int d\mathbf{p}_1 \ldots \int d\mathbf{p}_N \exp(-\beta H) = (2\pi mkT)^{3N/2},$$

where $H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m}$.

(b) Calculate the average internal energy from the derivative of the partition function

$$\langle U \rangle = -\left(\frac{\partial \ln Z(N,V,T)}{\partial \beta} \right)_V$$

Part II.

(a) Use Boltzmann statistics to calculate the partition function of the quantum harmonic oscillator with energy levels $\varepsilon_j = (j + \frac{1}{2})h\nu, \quad j = 0,1,2,..$

$$Z = \left(\frac{e^{-\theta/2T}}{1 - e^{-\theta/T}} \right),$$

where $\theta = h\nu/k$ is the characteristic temperature.

(b) Why can Boltzmann statistics be used for indistinguishable oscillators?

(c) Show that the Boltzmann entropy for a system of N oscillators is $S = -k\sum_j N_j \ln\left(\frac{N_j}{N}\right)$.

(d) Using the Boltzmann distribution show that the Boltzmann entropy becomes

$$S = \frac{U}{T} + kN \ln Z.$$

(e) Using $U = NkT^2\left(\frac{\partial}{\partial T} \ln Z\right)_V$ calculate the internal energy. Find the limiting behaviour of $U/Nk\theta$ as $T \to 0$ and $T \to \infty$. Graph the energy $U/Nk\theta$ as a function of T.

QUESTION 4 (20 marks)

(a) Photons in a cavity obey Bose-Einstein statistics. If the number of quantum states with frequencies in the range ν to $\nu + d\nu$ is

$$g(\nu)d\nu = \frac{8\pi V}{c^3}\nu^2 d\nu$$

show that the energy density at frequency ν is

$$u(\nu)d\nu = \frac{8\pi h V}{c^3}\left(\frac{\nu^3 d\nu}{e^{h\nu/kT} - 1}\right)$$

(b) Find the total energy density (energy per unit volume) by integrating over wavelength ($\lambda = c/\nu$). If the total energy density can be written as

$$\frac{U}{V} = aT^4$$

find the explicit expression for the constant a.

(c) Explain how the energy density as a function of wavelength given above (Planck's law) is related to the Rayleigh-Jeans law $u(\lambda)d\lambda \approx 8\pi kTV d\lambda/\lambda^4$, and to Wien's law

$$u(\lambda)d\lambda \approx V\left(\frac{8\pi hc}{\lambda^5}\right)e^{-hc/\lambda kT}d\lambda.$$

QUESTION 5 (20 marks)

(a) For a system of fermions where the density of states is given by

$$g(\varepsilon)d\varepsilon = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2}\varepsilon^{1/2}d\varepsilon.$$

Show that the Fermi energy at $T = 0$ is given by

$$\mu(0) = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3}$$

(b) The internal energy of a fermion gas is

$$U = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2}\int_0^\infty \frac{\varepsilon^{3/2}d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1}$$

Explain the interplay between the numerator and denominator of the integrand in determining the value of the internal energy.

(c) The electronic contribution to the internal energy is

$$U \approx \frac{3}{5}N\varepsilon_F\left(1+\frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2 - \ldots\right)$$

Find an expression for the electronic heat capacity.

(d) The internal energy can be written as an infinite sum with a set of undetermined coefficients

$$U = \frac{3}{5}NkT_F\sum_{i=0}^{\infty}a_i\left(\frac{T}{T_F}\right)^{2i},$$

where

$$T_F = \frac{h^2}{2mk}\left(\frac{3N}{8\pi V}\right)^{2/3}.$$

The dependence on T is explicit and T_F is a function of V. Show that the entropy is given by

$$S = \int_0^T \frac{1}{T'}\frac{\partial U}{\partial T'}dT' = \tfrac{3}{5}NkT_F\sum_{i=1}^{\infty}\frac{2i}{2i-1}a_i\left(\frac{T^{2i-1}}{T_F^{2i}}\right).$$

Show that the Helmholtz function is

$$F = U - TS = \tfrac{3}{5}NkT_F\left\{1 - \sum_{i=1}^{\infty}\frac{a_i}{2i-1}\left(\frac{T}{T_F}\right)^{2i}\right\}.$$

e) Hence, or otherwise, show that the relation $P = \tfrac{2}{3}(U/V)$ is exact for the fermion gas.