THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

MID-TERM EXAMINATION

APRIL 2014

PHYS3021 Statistical and Solid State Physics - PAPER 1

PHYS3020 Statistical Physics

Time Allowed – 50 minutes

Total number of questions - 2

Answer ALL questions

This exam is worth 7.5% of the final grade for PHYS3021 students

This exam is worth 15% of the final grade for PHYS3020 students

Candidates must supply their own university approved calculator.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.

This paper may be retained by the candidate

FORMULA SHEET

Boltzmann Entropy $S = k \ln W$

Statistics and Distributions

Boltzmann $W_B = N! \prod_{j=1}^{n} \frac{g_j^{N_j}}{N_j!}$ $\frac{N_j}{g_j} = \frac{N}{Z} e^{-\varepsilon_j/kT}$ $Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT}$

Maxwell-Boltzmann $W_{MB} = \prod_{j=1}^{n} \frac{g_j^{N_j}}{N_j!}$ $\frac{N_j}{g_j} = \frac{N}{Z} e^{-\varepsilon_j/kT}$ $Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT}$

Fermi-Dirac $W_{FD} = \prod_{j=1}^{n} \frac{g_j!}{N_j!(g_j - N_j)!}$ $\frac{N_j}{g_j} = \frac{1}{e^{(\varepsilon_j - \mu)/kT} + 1}$

Bose-Einstein $W_{BE} = \prod_{j=1}^{n} \frac{(N_j + g_j - 1)!}{N_j!(g_j - 1)!}$ $\frac{N_j}{g_j} = \frac{1}{e^{(\varepsilon_j - \mu)/kT} - 1}$

Microcanonical $f_{mc}(q,p) = \frac{\delta(H(q,p) - E)}{\int dq dp \delta(H(q,p) - E)}$

Canonical $f_C(q,p) = \frac{\exp(-\beta H(q,p))}{Z(N,V,T)}$ $Z(N,V,T) = \int dq dp \exp(-\beta H(q,p))$

Grand-canonical $f_G(q,p) = \frac{\exp(\beta(\mu N - H))}{\Xi(\mu,V,T)}$ $\Xi(\mu,V,T) = \sum_{N=0}^{\infty} z^N Z(N,V,T)$

Thermodynamic Potentials

Internal energy	U	$dU = TdS - PdV$
Enthalpy	$H = U + PV$	$dH = TdS + VdP$
Helmholtz function	$F = U - TS$	$dF = -SdT - PdV$
Gibbs function	$G = U - TS + PV$	$dG = -SdT + VdP$

Statistical Mechanics Canonical Ensemble

Internal energy $U = kT^2 \left(\frac{\partial \ln Z}{\partial T} \right)_V$ Pressure $P = kT \left(\frac{\partial \ln Z}{\partial V} \right)_T$

Mathematical identities

$$\ln N! \approx N\ln N - N \qquad \int_0^\infty \frac{x^3\,dx}{e^x-1} = \frac{\pi^4}{15} \qquad \frac{x}{5} = 1 - e^{-x} \quad \Rightarrow \quad x = 4.96$$

$$1 + y + y^2 + \ldots = \frac{1}{1-y} \qquad \int_{-\infty}^{\infty} dx e^{-x^2/\alpha} = \sqrt{\pi\alpha} \qquad \int_0^\infty d\varepsilon\, \varepsilon^{1/2} e^{-\varepsilon/\alpha} = \frac{\alpha}{2}\sqrt{\pi\alpha}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \frac{d}{dx}\sinh(x) = \cosh(x) \qquad \operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \frac{d}{dx}\cosh(x) = \sinh(x) \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)} \qquad \frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$$

$$\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)} \qquad \frac{d}{dx}\coth(x) = -\operatorname{csch}^2(x)$$

$$\int_{-\infty}^{\infty} \frac{e^y\,dy}{\left(e^y+1\right)^2} = 1 \qquad \int_{-\infty}^{\infty} \frac{y^2 e^y\,dy}{\left(e^y+1\right)^2} = \frac{\pi^2}{3}$$

QUESTION 1 (15 marks)

The thermodynamic probability W for a system of distinguishable particles, in which each level has an occupancy N_j and a degeneracy g_j, is

$$W = N!\prod_{j=0}^{n}\frac{g_j^{N_j}}{N_j!}$$

The two constraints are:

$$\sum_{j=1}^{n} N_j = N$$

$$\sum_{j=1}^{n} N_j E_j = U$$

(i) By minimising $ln\ W$ subject to these constraints, with an appropriate choice of Lagrange multipliers α and β, show that

$$N_j = g_j e^{\alpha+\beta E_j}$$

(6 marks)

(ii) By comparing the statistical definition of entropy with the thermodynamic definition, demonstrate that $\beta = -1/(k_B T)$. (4 marks)

(iii) Use the first constraint to determine e^{α}. (3 marks)

(iv) The expression above for N_j is called the Boltzmann distribution. In other physical situations we encounter the Maxwell-Boltzmann distribution, which looks formally the same. In what way is a system described by the Maxwell-Boltzmann distribution physically different from a system described by the Boltzmann distribution? (2 marks)

QUESTION 2 (15 marks)

Consider an ideal gas of particles of mass m. The Maxwell-Boltzmann distribution is applicable here. Each level has occupancy N_j and degeneracy g_j.

(i) The thermodynamic probability W for the Maxwell-Boltzmann distribution takes the form

$$W = \prod_{j=0}^{n} \frac{g_j^{N_j}}{N_j!}$$

Use this expression and the statistical definition of entropy to determine the entropy S as a function of the particle number N, partition function Z, and internal energy U. (5 marks)

(ii) The partition function Z for an ideal gas is:

$$Z = V\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}$$

where V is the volume. Using the known expressions for the pressure p and internal energy U in terms of Z, evaluate p and U explicitly. (5 marks) $P = kT/V$?

(iii) Use your results from (i) and (ii) to work out S, and show that the the entropy per mol, given by $s = S/n_m$ (n_m is the number of moles), satisfies

$$s = s_0 + R\ln V + c_v \ln T$$

Make sure you identify the constant s_0 and the molar heat capacity c_v . (5 marks)