THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

JUNE 2014

PHYS3021 Statistical and Solid State Physics - PAPER 1

PHYS3020 Statistical Physics

Time Allowed – 2 hours

Total number of questions - 5

Answer ALL questions

All questions ARE of equal value

This exam is worth 35% of the final grade for PHYS3021 students

This exam is worth 70% of the final grade for PHYS3020 students

Candidates must supply their own university approved calculator.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.

This paper may be retained by the candidate

FORMULA SHEET

Boltzmann Entropy $S = k\ln W$

Statistics and Distributions

Boltzmann $W_B = N!\prod_{j=1}^{n}\frac{g_j^{N_j}}{N_j!}$ $\frac{N_j}{g_j} = \frac{N}{Z}e^{-\varepsilon_j/kT}$ $Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT}$

Maxwell-Boltzmann $W_{MB} = \prod_{j=1}^{n}\frac{g_j^{N_j}}{N_j!}$ $\frac{N_j}{g_j} = \frac{N}{Z}e^{-\varepsilon_j/kT}$ $Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT}$

Fermi-Dirac $W_{FD} = \prod_{j=1}^{n}\frac{g_j!}{N_j!(g_j - N_j)!}$ $\frac{N_j}{g_j} = \frac{1}{e^{(\varepsilon_j-\mu)/kT}+1}$

Bose-Einstein $W_{BE} = \prod_{j=1}^{n}\frac{(N_j + g_j - 1)!}{N_j!(g_j - 1)!}$ $\frac{N_j}{g_j} = \frac{1}{e^{(\varepsilon_j-\mu)/kT}-1}$

Microcanonical $f_{mc}(q,p) = \frac{\delta(H(q,p)-E)}{\int dqdp\,\delta(H(q,p)-E)}$

Canonical $f_C(q,p) = \frac{\exp(-\beta H(q,p))}{Z(N,V,T)}$ $Z(N,V,T) = \int dqdp\exp(-\beta H(q,p))$

Grand-canonical $f_G(q,p) = \frac{\exp(\beta(\mu N - H))}{\Xi(\mu,V,T)}$ $\Xi(\mu,V,T) = \sum_{N=0}^{\infty} z^N Z(N,V,T)$

Thermodynamic Potentials

Internal energy	U	$dU = TdS - PdV$
Enthalpy	$H = U + PV$	$dH = TdS + VdP$
Helmholtz function	$F = U - TS$	$dF = -SdT - PdV$
Gibbs function	$G = U - TS + PV$	$dG = -SdT + VdP$

$P = -\left(\frac{\partial F}{\partial V}\right)$

Statistical Mechanics Canonical Ensemble

Internal energy $U = kT^2\left(\frac{\partial \ln Z}{\partial T}\right)_V$ Pressure $P = kT\left(\frac{\partial \ln Z}{\partial V}\right)_T$

$\mu = \partial F / \partial N$

$\left(\frac{\partial S}{\partial E}\right) = 1/T$

Mathematical identities

$$\ln N! \approx N \ln N - N \qquad \int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15} \qquad \frac{x}{5} = 1 - e^{-x} \quad \Rightarrow \quad x = 4.96$$

$$1 + y + y^2 + \ldots = \frac{1}{1-y} \qquad \int_{-\infty}^{\infty} dx e^{-x^2/\alpha} = \sqrt{\pi\alpha} \qquad \int_0^\infty d\varepsilon \varepsilon^{1/2} e^{-\varepsilon/\alpha} = \frac{\alpha}{2}\sqrt{\pi\alpha}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \frac{d}{dx}\sinh(x) = \cosh(x) \qquad \operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \frac{d}{dx}\cosh(x) = \sinh(x) \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)} \qquad \frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$$

$$\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)} \qquad \frac{d}{dx}\coth(x) = -\operatorname{csch}^2(x)$$

$$\int_{-\infty}^{\infty} \frac{e^y dy}{(e^y + 1)^2} = 1 \qquad \int_{-\infty}^{\infty} \frac{y^2 e^y dy}{(e^y + 1)^2} = \frac{\pi^2}{3}$$

QUESTION 1 (20 marks)

(a) Consider a system of three distinguishable particles that can have energies $n\varepsilon$, where $n \geq 0$ is an integer. The levels are all nondegenerate ($g_j = 1$) and the total energy of the system is constrained to be $U = 7\varepsilon$. Tabulate all the possible microstates and label all the possible macrostates.

(b) Repeat the above for three indistinguishable particles obeying Bose statistics.

(c) Repeat the above for three indistinguishable particles obeying Fermi statistics. What is the average occupancy of the fermion system?

(d) In how many ways can the energy of the fermion system be increased to $U = 8\varepsilon$? In how many ways can the energy of the boson system be increased to $U = 8\varepsilon$?

(e) How many macrostates exist for three fermions with energy $U = 8\varepsilon$?

(f) What is the change in entropy of the fermion system as the energy is increased from $U = 7\varepsilon$ to $U = 8\varepsilon$?

check

(g) If the change is isothermal, what is the temperature of the system?

QUESTION 2 (20 marks)

Photons satisfy $\omega = ck$, where ω is the angular frequency, c the speed of light and k the wave vector.

(a) What is the chemical potential of photons and why? A simple sentence will be enough.

(b) Consider a photon gas in a cubic box of side L. Write down the density of states $g(k)\,dk$ of photons in this box in terms of $V = L^3$. Do not forget the several possible polarizations.

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$g(k)\,dk = \text{const.}\, k^2 dk$

(c) Ignoring the zero-point energy, a simple harmonic oscillator has energy

$$u_{SHO}(\omega) = \frac{\hbar\omega}{e^{\hbar\omega/k_BT} - 1}$$

We can treat each photon as a simple harmonic oscillator ignoring the zero point energy. Why is this energy being left out? Again, a simple sentence is sufficient.

(d) Determine the internal energy U of the photon gas. You may find the following integral useful:

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$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$$

x^3

(e) The internal energy can be written as the integral over ω of an energy density u_ω. Write down u_ω and the corresponding u_λ as a function of wavelength. What are u_ω and u_λ at low frequencies, $\hbar\omega \ll k_BT$? Use your expression for u_λ in this limit to explain in two sentences the ultraviolet catastrophe.

QUESTION 3 (20 marks)

Consider a gas of spin-1/2 fermions at T = 0.

(a) What is the relationship between the Fermi energy E_F and the chemical potential at $T = 0$?

(b) Sketch the Fermi-Dirac distribution function at $T = 0$.

(c) Determine an expression for E_F in terms of the particle number density n.

(d) Express the average energy per particle in terms of E_F.

(e) What is the relationship between pressure and energy density u? Use this to evaluate the electron degeneracy pressure exerted by the electrons in a star. Express your result in terms of the electron number density n_e.

QUESTION 4 (20 marks)

Consider a degenerate electron gas in a magnetic field $B \parallel z$. The magnetic moment μ due to its spin gives rise to an interaction with the magnetic field of the form $-\boldsymbol{\mu}.\boldsymbol{B}$, and $\mu B << E_F$, where E_F is the Fermi energy. Denote by n_+, n_- the densities of electrons that have their magnetic moment antiparallel and parallel to **B** respectively. Show that the magnetic susceptibility in the low-temperature regime is given by

$$\chi = \frac{3n\mu^2}{2E_F}.$$

QUESTION 5 (20 marks)

Consider a gas of particles with non-zero mass in the ultrarelativistic limit $E = pc$, where p is the momentum and c the speed of light.

(a) Write down an expression for the density of states $g(k)$.

(b) Write down an expression for the single-particle partition function Z_1. Using $g(k)$ from (a), evaluate the single-particle partition function and show that it can be written as:

$$Z_1 = \frac{V}{\Lambda^3}.$$

What is Λ? You may find the following integral useful:

$$\int_0^\infty x^n e^{-x} = n!$$

(c) What is the N-particle partition function Z_N in terms of Z_1 for a gas of indistinguish-able particles?

(d) Use your results from (a) - (c) to determine the internal energy U, heat capacity C_V, Helmholtz function F, and pressure p for an ultrarelativistic gas of indistinguishable particles. Express the pressure p in terms of the internal energy density $u = U/V$.