THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION: JUNE 2014

PHYS3011 & PHYS3230
Electrodynamics

Time allowed – 2 hours
Total number of questions – 5
Answer ALL 5 questions

Answer Part 1 (Questions 1 and 2) in one answer book

and Part 2 (Questions 3, 4, 5) in a separate answer book

All questions are of equal value

Candidates must supply their own, university-approved calculator.

All answers must be written in ink.
Except where they are expressly required, pencils may only be used for drawing, sketching, or graphical work.

Candidates may keep this question paper.

Part 1

Question 1 (20 marks)

Consider an electromagnetic wave propagating in vacuum (i.e., no sources).

(a) Explain the meaning of these equations

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{S} = 0, \tag{1}$$

$$u \equiv \frac{\epsilon_0}{2}|\vec{E}|^2 + \frac{1}{2\mu_0}|\vec{B}|^2, \tag{2}$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} \tag{3}$$

for the system. Be sure to explain the meaning of each quantity appearing in the equations.

(b) Show that equation (1) can be derived from Maxwell's equations.

(c) Suppose the electromagnetic wave under consideration is a plane wave with wave vector $\vec{k}$. Write down the corresponding $\vec{E}$ and $\vec{B}$ fields, and evaluate $\vec{S}$ explicitly.

Question 2 (20 marks)

(a) Consider the linear transformation

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}.$$

What are the conditions on the matrix Λ for this to be a Lorentz transformation? Derive an expression for the inverse transform Λ^{-1} in terms of Λ^T, the transpose of Λ.

(b) Show that the continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0,$$

can be derived from the field equation $\partial_\mu F^{\nu\mu} = \mu_0 J^\nu$.

(c) Suppose an inertial reference frame S' is moving away from a frame S with velocity v in the positive x_1 direction. If the observer in S measures a static charge distribution

$$\rho(\vec{x}) = Q \exp\left(-\frac{x^i x_i}{a}\right),$$

where $i = 1, 2, 3$, find the charge and current distributions as measured by the observer in S'. Discuss the nonrelativistic limit of your result.

(Continued overleaf)

Part 2

Remember to use a new answer book for questions in Part 2.

Question 3 (20 marks)

The wave equation for a plane wave moving in the positive x-direction in a conducting medium is given in the formula sheet. The solution is also given, and the equation for k in a good conductor.

(a) By substituting the given solution in the wave equation, find the general equation for k, and show it reduces to the given form in a good conductor.

(b) If $k = \alpha + i\beta$, find the values of α and β in terms of μ_0, ϵ_0, σ and ω. $\left[\text{NB } \sqrt{i} = \frac{(1+i)}{\sqrt{2}}\right]$

(c) Find the limiting values of α and β for a very good conductor and for a very poor conductor.

(d) Find the skin depth in the conducting medium.

(e) Find the refractive index of the conducting medium.

Question 4 (20 marks)

(a) Explain the terms "retarded potential" and "Hertzian dipole".

The $\mathbf{E}$ field from a short oscillating electric dipole, $p = p_0 \cos \omega t$, at the origin and aligned along the z axis, is given by

$$\mathbf{E}(r, \theta, t) = -\frac{\mu_0 p_0 \omega^2 \sin\theta}{4\pi r} \cos[\omega(t - r/c)]\, \hat{\theta},$$

where r is large compared to the size of the dipole, and to the wavelength of the radiation.

(b) Using the plane-wave approximation, valid for large r, write down the corresponding expression for $\mathbf{B}$ (both magnitude and direction). (Note that the direction of propagation, $\hat{\mathbf{k}}$, is along $\hat{\mathbf{r}}$.)

(c) Hence, (i) find the Poynting vector for this radiation, and (ii) the total power radiated in all directions.

(Continued overleaf)

Question 5 (20 marks)

(a) Starting from the boundary conditions for **E**, **D**, **B** and **H** at the boundary of two dielectrics, and remembering that E $= v$B in a dielectric, show that the ratio of the reflected power to the incident power for a wave at normal incidence from a vacuum onto a medium of refractive index n and permeability $\mu = \mu_0$ is

$$R = \left(\frac{n-1}{n+1}\right)^2,$$

as given on the formula sheet.

(b) An observer, wearing polarising glasses to completely absorb horizontally polarised light, is looking at light reflected from the surface of the sea at an angle of 45°. Calculate the ratio of light intensity he sees with and without these glasses. (NB this is not the ratio to the intensity of the incident light.) The refractive index of water is 1.33.

Useful Formulae: PHYS3011/PHYS3230

$\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{Fm^{-1}} \quad \mu_0 = 4\pi \times 10^{-7}\ \mathrm{Hm^{-1}} \quad c = 3 \times 10^8\ \mathrm{ms^{-1}}$

Volume element $= dx\,dy\,dz \;=\; r^2 \sin\theta\, dr\, d\theta\, d\phi$

Surface area of sphere $= 4\pi r^2$ Volume of sphere $= \frac{4}{3}\pi r^3$

Divergence Theorem: $\int_V \nabla\cdot\mathbf{A}\, dV = \int_S \mathbf{A}\cdot d\mathbf{S}$ (S is the surface enclosing V)

Stokes' Theorem: $\int_S (\nabla\times\mathbf{A})\cdot d\mathbf{S} = \oint_L \mathbf{A}\cdot d\mathbf{l}$ (L is the curve bounding S)

Vector identity: $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\,\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\,\mathbf{C}$

So: $\nabla\times(\nabla\times\mathbf{E}) = \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E}$

Also: $\nabla\cdot(\mathbf{E}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{B})$

∇^2 in spherical polar coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Electrostatics:

Charge conservation: $I = -\frac{dq}{dt}$

$q = \int_V \rho\, dV$ (charge density) $\quad I = \int_S \mathbf{J}\cdot d\mathbf{S} = \int_V \nabla\cdot\mathbf{J}\, dV$ (current density)

$\therefore\ \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}$ (equation of continuity)

E field defined by: $\mathbf{F}_E = q\,\mathbf{E}$

Coulomb's Law: $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}}$

Gauss's Law: $\Phi_E = \int_S \mathbf{E}\cdot d\mathbf{S} = \frac{q}{\epsilon_0} = \frac{1}{\epsilon_0}\int_V \rho\, dV$

$\therefore\ \nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}$ (Gauss's Law)

[Also, since **E** is conservative in electro*statics*: $\nabla\times\mathbf{E} = 0$]

Magnetism:

B field defined by: $\mathbf{F}_B = q\,\mathbf{v}\times\mathbf{B}$ ie $d\mathbf{F} = dq\,\mathbf{v}\times\mathbf{B} = I d\mathbf{l}\times\mathbf{B}$

No magnetic monopoles: $\nabla\cdot\mathbf{B} = 0$

Biot-Savart Law: $d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I' d\mathbf{l}'\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\frac{dq'}{r^2}\,\mathbf{v}'\times\hat{\mathbf{r}}$

Ampère's Law: $\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0(N)I = \mu_0\int_S \mathbf{J}\cdot d\mathbf{S}$

(where I = current enclosed)

$\therefore\ \nabla\times\mathbf{B} = \mu_0\mathbf{J}$ (Ampère's Law)

Faraday's Law: $\mathcal{E} = \oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}$

$\therefore\ \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$ (Faraday's Law)

Dielectric materials:

$\mathbf{P} = \chi\epsilon_0\mathbf{E} \qquad \mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = (1+\chi)\epsilon_0\mathbf{E} = \epsilon_r\epsilon_0\mathbf{E}$

E field (and potential difference) is reduced by a factor ϵ_r in the bulk.

Energy density, $u = \frac{1}{2}\epsilon_r\epsilon_0 E^2$ per unit volume $= \frac{1}{2}\mathbf{E}\cdot\mathbf{D}$

Gauss's Law for **D** : $\int \mathbf{D}\cdot d\mathbf{S} = q_{free} \quad \nabla\cdot\mathbf{D} = \rho_{free}$

At a boundary, $E_\parallel$ and V are continuous ($D_\perp$ is continuous.)

Cavities in dielectrics: $\mathbf{E}_{\rm local} = \mathbf{E}_{\rm bulk}$ for a needle-shaped cavity;

$\mathbf{E}_{\rm local} = \mathbf{E}_{\rm bulk} + \mathbf{P}/\epsilon_0$ for a disc-shaped cavity;

$\mathbf{E}_{\rm local} = \mathbf{E}_{\rm bulk} + \mathbf{P}/3\epsilon_0$ for a spherical cavity.

Clausius-Mossotti equation: $\frac{n\alpha}{3\epsilon_0} = \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)$

Capacitance:

stored charge $Q = C\Delta V$ [C]
stored energy $U = \frac{1}{2}Q\Delta V = \frac{1}{2}C(\Delta V)^2$ or $\frac{1}{2}Q^2/C$ [J]

capacitance of parallel-plate capacitor is $C = \epsilon_r\epsilon_0 A/d$ [F]

capacitance of isolated sphere is $C = 4\pi\epsilon_r\epsilon_0 R$ [F]

DC Circuits:

Ohm's Law: $\Delta V = IR$ resistance, $R = \rho l/A$ $[\Omega]$

Kirchhoff's Laws: (1) $\Sigma I = 0$ at a junction
(2) $\Sigma\mathcal{E} - \Sigma IR = 0$ around each loop

Joule heating: power dissipated, $P = I\Delta V = I^2R = (\Delta V)^2/R$ [W]

Ohm's law: $\mathbf{J} = \sigma\mathbf{E}$ power dissipated/unit volume $= \mathbf{J}\cdot\mathbf{E} = \sigma E^2$

Magnetic media:

$\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}) = \mu_0(1+\chi_m)\mathbf{H} = \mu_r\mu_0\mathbf{H}$ *ie* $\mathbf{H} = \dfrac{\mathbf{B}}{\mu_0} - \mathbf{M}$

$\nabla\cdot\mathbf{B} = 0$, so $\nabla\cdot\mathbf{H} + \nabla\cdot\mathbf{M} = 0$

Ampère's law becomes: $\nabla\times\mathbf{H} = \mathbf{J}_{free}$

At a boundary, $B'_\perp = B_\perp$ and $H'_\parallel = H_\parallel$

Inductance:

Mutual inductance: $\Phi_1 = L_{12}I_2$, $\Phi_2 = L_{12}I_1$, Self inductance: $\Phi = LI$

Self Inductance of a solenoid: $L = \mu_r\mu_0\dfrac{N^2}{\ell}A$ magnetic energy: $U = \frac{1}{2}LI^2$

Energy density in magnetic field: $u = \frac{1}{2}\dfrac{B^2}{\mu_r\mu_0} = \frac{1}{2}\mathbf{B}\cdot\mathbf{H}$

Maxwell's Equations

In a vacuum:

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

Lorentz force law: $\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$

Maxwell's equations in dielectric and magnetic media:

$$\nabla\cdot\mathbf{D} = \rho \qquad \nabla\times\mathbf{E} = -\dot{\mathbf{B}}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{H} = \mathbf{J} + \dot{\mathbf{D}}$$

EM Waves:

Wave equation for $\mathbf{E}$ in free space: $\nabla^2\mathbf{E} = \mu_0\epsilon_0\dfrac{\partial^2\mathbf{E}}{\partial t^2}$ *ie* $c = 1/\sqrt{\mu_0\epsilon_0}$

(in a medium: $v = 1/\sqrt{\mu_r\mu_0\epsilon_r\epsilon_0} = c/n$, n = refractive index)

Solution: $E = E_0\sin(kx - \omega t)$ for monochromatic wave travelling in +ve x-direction.

$\mathbf{E}$, $\mathbf{B}$ and the direction of propagation $\hat{\mathbf{k}}$ are mutually perpendicular:

$\hat{\mathbf{k}}\cdot\mathbf{E} = 0$ $\hat{\mathbf{k}}\cdot\mathbf{B} = 0$ $c\mathbf{B} = \hat{\mathbf{k}}\times\mathbf{E}$ $\hat{\mathbf{E}}\times\hat{\mathbf{B}} = \hat{\mathbf{k}}$

The direction of $\mathbf{E}$ is the direction of polarization of the E-M wave.

Impedance of free space, $Z_0 = \dfrac{|E|}{|H|} = \sqrt{\dfrac{\mu_0}{\epsilon_0}} = 377\,\Omega$

Poynting vector: $\mathbf{N}$ (or $\mathbf{S}$) $= \mathbf{E}\times\mathbf{H} = \dfrac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = E^2/Z_0 = H^2 Z_0$

NB: Wave number, $k = \dfrac{2\pi}{\lambda}$ Angular frequency, $\omega = 2\pi f$

Phase velocity, $v_p = f\lambda = \dfrac{\omega}{k}$ Group velocity $v_g = \dfrac{d\omega}{dk}$

Malus' Law: $I(\theta) = I(0)\cos^2\theta$ for polarizers at relative angle θ.

Reflection and Refraction at interface between two dielectrics

Reflection: $\theta_r = \theta_i$

Refraction: $n_2\sin\theta_t = n_1\sin\theta_i$ (Snell's Law)

Critical angle: $\sin\theta_i = n_2/n_1$ if $n_1 > n_2$

Fresnel Equations ($\mu_r = 1$)

For $\mathbf{E}$ *parallel to the plane of incidence*

$$r_\parallel = \frac{E_r}{E_i} = \frac{(n_2/n_1)\cos\theta_i - \cos\theta_t}{(n_2/n_1)\cos\theta_i + \cos\theta_t} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

$$t_\parallel = \frac{E_t}{E_i} = \frac{2\cos\theta_i}{(n_2/n_1)\cos\theta_i + \cos\theta_t}$$

For $\mathbf{E}$ *perp to the plane of incidence*

$$r_\perp = \frac{E_r}{E_i} = \frac{\cos\theta_i - (n_2/n_1)\cos\theta_t}{\cos\theta_i + (n_2/n_1)\cos\theta_t} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

$$t_\perp = \frac{E_t}{E_i} = \frac{2\cos\theta_i}{\cos\theta_i + (n_2/n_1)\cos\theta_t}$$

If $n = n_2/n_1$:

$r_{\parallel} = 0$ at the Brewster angle, $\theta_i = \theta_B$, where $\tan\theta_B = n$

At normal incidence $(\theta_i = 0)$, reflecting power, $R_0 = r_{\parallel}^2 = r_{\perp}^2 = \left(\frac{n-1}{n+1}\right)^2$

$$T_0 = \frac{Z_1}{Z_2} t^2 = nt^2 = \frac{4n}{(1+n)^2} \qquad (R+T) = 1$$

EM waves in a conducting medium

Wave equation: $\frac{\partial^2 \mathbf{E}}{\partial x^2} = \mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t} + \epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$

Solution: $E_y = E_0 \exp i(kx - \omega t)$ with $k^2 = i\omega\mu_0\sigma$ (k is complex)

Skin depth, $\delta = \sqrt{\left(\frac{2}{\mu_0 \sigma \omega}\right)}$ Effective (complex) refractive index $n = \frac{kc}{\omega}$

Reflection from metal at normal incidence $r = \frac{E_r}{E_i} = \left(\frac{1-n}{1+n}\right) \approx -1$

Transmission Lines
(here L and C are the inductance and capacitance per unit length)

Wave velocity, $v = \frac{1}{\sqrt{LC}}$ Characteristic impedance, $Z_0 = \sqrt{\frac{L}{C}}$

Twin wires, separation b, each of radius a, $C = \frac{\pi\epsilon_r\epsilon_0}{\ln(b/a)}$ $L = \frac{\mu_r\mu_0}{\pi}\ln(b/a)$

for air/vacuum: $Z_0 = 120\ln(b/a)\ \Omega$

Co-axial cable, radii a (inner) and b (outer), $C = \frac{2\pi\epsilon_r\epsilon_0}{\ln(b/a)}$ $L = \frac{\mu_r\mu_0}{2\pi}\ln(b/a)$

for air/vacuum: $Z_0 = 60\ln(b/a)\ \Omega$

Stripline: two conductors of width b, separation a, $C = \frac{\epsilon_r\epsilon_0 b}{a}$ $L = \frac{\mu_r\mu_0 a}{b}$

for air/vacuum: $Z_0 = 377\,\frac{a}{b}\ \Omega$

Waveguide equation $\frac{1}{\lambda_g^2} = \frac{1}{\lambda_0^2} - \frac{1}{(2b)^2}$

Lorentz transformation equations for $\mathbf{E}$ and $\mathbf{B}$:

If O' is moving with speed $+V$ along x axis of O:

$$E'_x = E_x \qquad E'_y = \gamma(E_y - VB_z) \qquad E'_z = \gamma(E_z + VB_y)$$

$$B'_x = B_x \qquad B'_y = \gamma(B_y + \frac{V}{c^2}E_z) \qquad B'_z = \gamma(B_z - \frac{V}{c^2}E_y)$$

$B^2 - \frac{E^2}{c^2}$ and $\mathbf{E}\cdot\mathbf{B}$ are invariants.

Potentials:

Inside a solenoid of radius R, at distance a from the axis: $\mathbf{A} = \frac{\mu_0 n I a}{2}\,\hat{\phi}$

Outside the solenoid, $\mathbf{A} = \frac{\mu_0 n I R^2}{2a}\,\hat{\phi}$

$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla \times \mathbf{A}$.

Lorentz Gauge: $\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$

$$\left(\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \nabla^2 V\right) = \frac{\rho}{\epsilon_0} \qquad \left(\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A}\right) = \mu_0 \mathbf{J}$$

Liénard-Wiechert potentials (from a moving charge):

$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{(Rc - \mathbf{R}\cdot\mathbf{v})}$ and $\mathbf{A}(\mathbf{r},t) = \frac{\mathbf{v}}{c^2}V(\mathbf{r},t)$ where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$

Electric Dipole Radiation:

$$V(\mathbf{r},t) = \frac{-p_0\,\omega\cos\theta}{4\pi\epsilon_0 rc}\sin\omega t' \qquad \mathbf{A}(\mathbf{r},t) = \frac{-\mu_0\,p_0\,\omega}{4\pi r}\sin\omega t'\ \hat{\mathbf{z}}$$

Magnetic Dipole Radiation:

$$V(\mathbf{r},t) = 0 \qquad \mathbf{A}(\mathbf{r},t) = \frac{-\mu_0\,m_0\,\omega\sin\theta}{4\pi rc}\sin\omega t'\ \hat{\phi}$$

NB $t' = t - |\mathbf{r} - \mathbf{r}'|/c$ (Retarded time.)

In both cases, $\mathbf{E}$ is $\perp$ to $\mathbf{B}$, and both are $\perp$ to $\mathbf{r}$, and $E = cB$.

Lorentz transformation for an inertial frame S' moving with speed v in the x^1 direction in the inertial frame S.

$$\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Invariant interval and the Minkowski metric

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4-velocity and 4-momentum

$$u^\mu \equiv \frac{dx^\mu}{d\tau} = \gamma\,(c, \vec{v}),$$

$$p^\mu \equiv m u^\mu = \left(\frac{E}{c}, \vec{p}\right)$$

Relativistic electrodynamics

$$J^\mu = \left(\rho c, \vec{J}\right) \qquad \partial_\mu J^\mu = 0$$

$$A^\nu = \left(\frac{\varphi}{c}, \vec{A}\right) \qquad \frac{dp^\mu}{d\tau} = qF^{\mu\nu} u_\nu$$

$$\partial_\mu F^{\nu\mu} = \mu_0 J^\nu$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} = -F^{\nu\mu}$$

$$\Box^2 \equiv \partial_\mu \partial^\mu = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$