**Question 1.** Heisenberg's uncertainty principle (Marks 40).

a. Formulate briefly the Heisenberg uncertainty principle. Outline, also very briefly its relation with the de Broglie relations.

b. Assume that a particle of mass $m$ propagates along the $x$-axis in the potential

$$U(x) = \frac{k\,x^{2n}}{2n} \qquad (1)$$

where $k$ is a positive constant and $n \geq 1$ is an integer. Using Heisenberg's uncertainty principle, estimate the ground state energy $E$, as well as the averaged kinetic and potential energies in this state.

Hint. To simplify algebraic calculations one can choose units $\hbar = m = 1$. Then the only dimensional parameter left is $k$. If you still struggle with the algebra, keep in mind that up to a numerical factor the dependence of the energy on $k$ (and $m$) can be recovered from simple dimensional analyses.)

**Question 2.** Quantum oscillator (Marks 60).
Consider the conventional quantum oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\,\omega^2 x^2}{2} \qquad (2)$$

a. Write down the expression for its energy spectrum $E_n$.

Introduce the creation and annihilation operators

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{x}{b} + b\frac{d}{dx}\right) \qquad (3)$$

$$\hat{a}^{+} = \frac{1}{\sqrt{2}}\left(\frac{x}{b} - b\frac{d}{dx}\right)$$

where $b = \sqrt{\hbar/m\,\omega}$ is the magnetic length (for the following calculations it can be convenient to choose units in which $b = 1$).

b. Prove that the Hamiltonian (2) can be rewritten as follows

$$\hat{H} = \hbar\,\omega\,(\hat{a}^{+}\hat{a} + \tfrac{1}{2}) \qquad (4)$$

Hint: remember the commutation relation

$$[\hat{a}, \hat{a}^{+}] = 1 \qquad (5)$$

c. Find an explicit expression for the ground state wave function $\psi_0(x)$ of the Harmonic oscillator.
Hint: Remember that Eq.(4) allows one to write the linear first order differential equation on $\psi_0(x)$, which solution is straightforward.

d. Consider the Hamiltonian

$$\hat{H}_1 = \hbar\,\omega(\hat{a}^{+}\hat{a} + \lambda\,\hat{a} + \lambda^{*}\hat{a}^{+}). \qquad (6)$$

where $\lambda$ is a complex-valued constant. Find the energy spectrum of $\hat{H}_1$.