UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

NOVEMBER 2015

PHYS2210 Electromagnetism and Thermal Physics (Paper 1)

PHYS2050 Electromagnetism

Time Allowed – 2 hours

Total number of questions – 4

All questions are of equal value

Answer all questions

Candidates must supply their own, university approved calculator.

Candidates may bring one A4 page of notes, handwritten on both sides. This must be returned with your exam book.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.

Candidates may keep this paper.

CARTESIAN: $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $\quad d\mathbf{r} = dx\,dy\,dz$

Gradient $$\nabla T = \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

Divergence $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl $$\nabla \times \mathbf{v} = \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

Laplacian $$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

SPHERICAL: $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\, d\phi\,\hat{\boldsymbol{\phi}}$; $\quad d\mathbf{r} = dl_r dl_\theta dl_\phi = r^2 \sin\theta\, dr d\theta d\phi$

Gradient $$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian $$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}$$

CYLINDRICAL: $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $\quad d\mathbf{r} = r dr\, d\phi\, dz$

Gradient $$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

Divergence $$\nabla \cdot \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}\left(r v_r\right) + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl $$\nabla \times \mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\boldsymbol{\phi}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_\phi) - \frac{\partial v_r}{\partial \phi}\right)\hat{\mathbf{z}}$$

Laplacian $$\nabla^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$

TRIPLE PRODUCTS

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

PRODUCT RULES

(3) $\nabla(fg) = f\nabla g + g\nabla f$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

SECOND DERIVATIVES

(9) $\nabla\cdot(\nabla\times\mathbf{v}) = 0$

(10) $\nabla\times(\nabla T) = 0$

(11) $\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$

FUNDAMENTAL THEOREMS

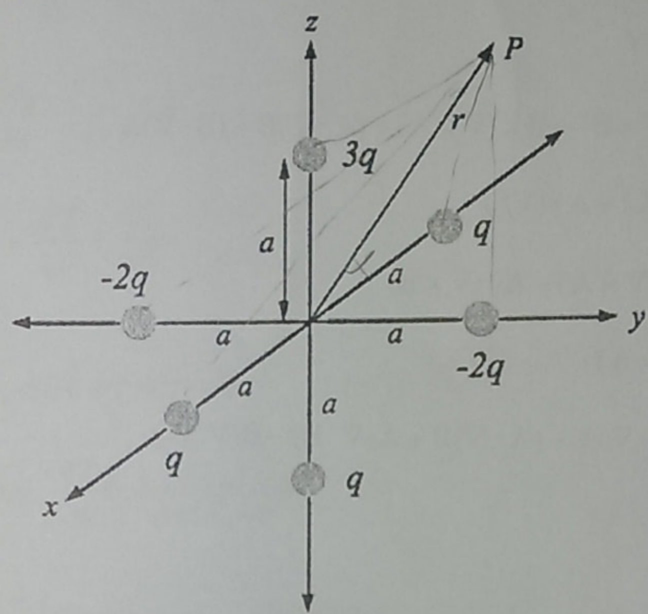
Gradient Theorem $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T)\cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a})$

Divergence Theorem $\int_{volume} (\nabla\cdot\mathbf{v})\,d\mathbf{r} = \oint_{surface} \mathbf{v}\cdot d\mathbf{a}$

Curl Theorem $\int_{surface} (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \oint_{boundary\ line} \mathbf{v}\cdot d\mathbf{l}$

Question 1 (25 marks)

Consider six point charges arranged as shown below.



(a) Calculate (i) the monopole moment and (ii) the dipole moment of this charge distribution.

(b) Use your results from (a) to show that the electric potential due to this charge distribution at a point P very far away from the origin (i.e., $r \gg a$) is given approximately by

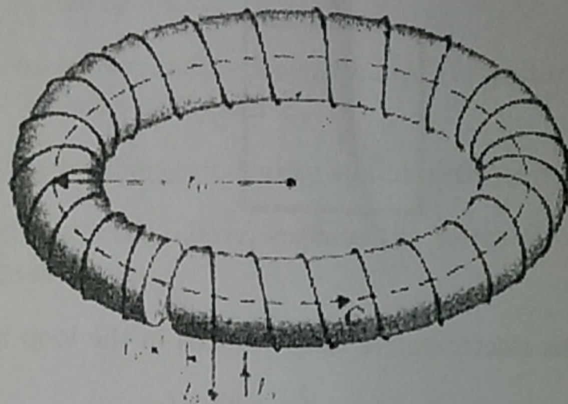
$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0 r}\left[2 + \frac{2a\cos\theta}{r} + \ldots\right],$$

where $\cos\theta \equiv \hat{z}\cdot\hat{r}$.

(c) Calculate the corresponding electric field and express your answer in spherical coordinates.

Question 2 (25 marks)

Suppose that N turns of wire are tightly wound around a toroidal core of a ferromagnetic material with permeability μ. The core has a mean radius r_0, a circular cross section of radius a, and a narrow air gap of length ℓ_g, as shown below. A steady current I_0 flows in the wire.



(a) Using the appropriate Maxwell equation, show that the magnetic field inside the ferromagnetic material, $\vec{B}_f$, and the magnetic field in the air gap, $\vec{B}_g$, satisfy

$$\vec{B}_f = \vec{B}_g = B\hat{\phi},$$

where B is a constant real number and $\hat{\phi}$ is the angular direction around the toroid. You may assume $r_0 \gg a$ so that to first approximation the magnetic field does not depend on the distance from the centre of the toroid.

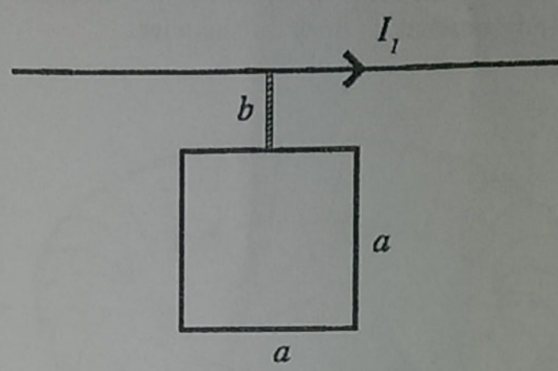
(b) Using the contour C shown above, write down an expression relating the $\vec{H}$ field inside the ferromagnetic material, $\vec{H}_f$, and its counterpart in the air gap, $\vec{H}_g$, to the number of turns N and the current I_0.

(c) Use your result from (b) to show that

$$\vec{B} = \frac{\mu\mu_0 N I_0}{\mu_0(2\pi r_0 - \ell_g) + \mu\ell_g}\hat{\phi}.$$

Question 3 (25 marks)

Consider an infinitely long straight wire carrying a time-dependent current $I_1(t)$. Hanging off the wire is a rigid square wire loop of side length a.



(a) Show that the electromotive force induced in the loop is given by

$$\mathcal{E} = \frac{\mu_0 a}{2\pi} \ln\left(\frac{a+b}{b}\right) \frac{dI_1}{dt}.$$

(b) Suppose the current $I_1(t)$ increases with time. What is the direction of the current induced in the loop? Please justify you answer.

(c) Show that the magnitude of the induced electric field in the square loop is

$$E = \frac{\mu_0}{8\pi} \ln\left(\frac{a+b}{b}\right) \frac{dI_1}{dt}.$$

(d) Calculate the magnitude of the induced current in the square loop, assuming that the loop is made of a material with conductivity σ and has a cross-sectional area A.

Question 4 (25 marks)

Consider an electromagnetic wave propagating in free space. Its electric field component is given by

$$\vec{E}(x,t) = E_y \sin(kx - \omega t)\ \hat{y} + E_z \cos(kx - \omega t)\ \hat{z}$$

(a) Demonstrate explicitly that this expression satisfies all four of Maxwell's equations in free space.

(b) Compute the corresponding magnetic field (magnitude and direction). Express it in terms of E_y and E_z.

(c) Compute the total energy density stored in the wave.

(d) Using the expression above, explain the terms (i) circular and (ii) elliptical polarisation.