(1) a) This system is equivalent to a damped, forced harmonic oscillator, with
$m \rightarrow L$, $b \rightarrow R$, $k \rightarrow \frac{1}{C}$, $F_0 \rightarrow V_0$.

$$\therefore \quad \omega_0 = \sqrt{\frac{k}{m}} \rightarrow \sqrt{\frac{1}{LC}} = \left(50\times10^{-3} \times 20\times10^{-6}\right)^{-\frac{1}{2}}$$

$$= \underline{1000} \text{ rad. s}^{-1} \qquad \text{(Answer)}$$

$$\gamma = \frac{b}{2m} \rightarrow \frac{20}{2\times50\times10^{-3}} = \underline{200} \text{ s}^{-1} \qquad \text{(Answer)}$$

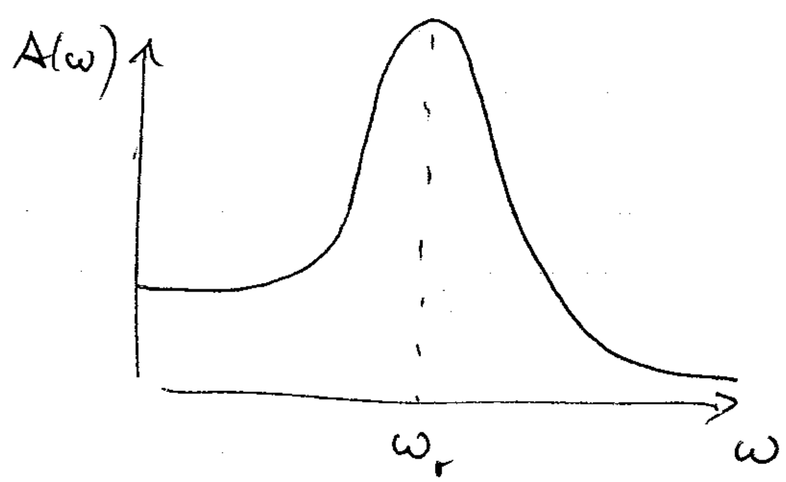
$\therefore$ $\omega_0 >> \gamma$: the system is underdamped.

b) Resonant frequency $\omega_r = \sqrt{\omega_0^2 - 2\gamma^2}$

$$= \sqrt{[10^6 - 4\times10^4]} = 9.8 \times 10^2 \text{ rad. s}^{-1} \qquad \text{(Answer)}$$

c) $$Q = \frac{\omega_0}{2\gamma} = \frac{1000}{400} = \underline{2.5}. \qquad \text{(Answer)}$$

d)



$$A(0) = \frac{F_0}{\sqrt{m^2\omega_0^4}} \rightarrow \frac{V_0}{L\omega_0^2} = V_0 C$$

$$= 50 \times 20 \times 10^{-6} = \underline{10^{-3}\text{ V}}. \qquad \text{(Answer)}$$

e) System is critically damped if

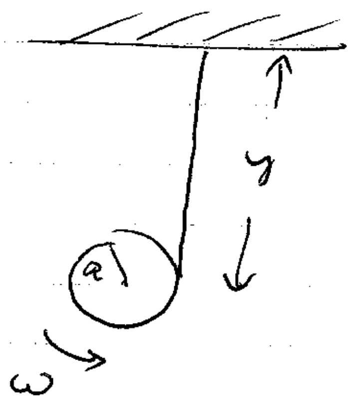
$\omega_0 = \gamma$ i.e. $\sqrt{\frac{k}{m}} = \frac{b}{2m}$

or $b^2 = 4mk$

$\rightarrow$ $R^2 = \frac{4L}{C}$.

$\therefore$ $R = \sqrt{\frac{4 \times 50 \times 10^{-3}}{20 \times 10^{-6}}} = \underline{100\ \Omega}$ (Answer).

(2) a)



Kinetic energy : $T = \frac{1}{2} m\dot{y}^2 + \frac{1}{2} I\omega^2$

$$\omega a = \dot{y} \quad , \quad I = \frac{2}{5} m a^2$$

$$\therefore T = \frac{7}{10} m\dot{y}^2$$

Potential energy : $V = -mgy$

$$\therefore L = T - V = \frac{7}{10} m\dot{y}^2 + mgy$$

b) Lagrange eq'n. of motion :

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0$$

$$\therefore \quad mg - \frac{d}{dt}\left(\frac{7}{5} m\dot{y}\right) = 0$$

$$\therefore \quad \ddot{y} = \frac{5}{7} g \qquad \boxed{\text{QED}}$$

c) Momentum conjugate to y is :

$$p_y = \frac{\partial L}{\partial \dot{y}} = \frac{7}{5} m\dot{y} \qquad (\text{N.B.! } \underline{\text{Not}}\ m\dot{y}) \qquad (1)$$

$$\therefore \quad H = p_y \dot{y} - L = \frac{7}{10} m\dot{y}^2 - mgy$$

$$(= T + V).$$

$$= \frac{5}{14} \frac{p_y^2}{m} - mgy \qquad (\text{N.B.}) \qquad (2)$$

using (1).

d) Hamiltonian eq'ns. of motion:

$$\frac{\partial H}{\partial p} = \dot{y} \quad \text{i.e.} \quad \frac{5p}{7m} = \dot{y} \quad \Rightarrow \quad p = \frac{7}{5} m\dot{y}$$

$$\frac{\partial H}{\partial y} = -\dot{p} \quad \text{i.e.} \quad -mg = -\frac{7}{5} m\ddot{y}$$

$$\text{i.e.} \quad \ddot{y} = \frac{5}{7} g \qquad \boxed{QED}$$

(3) a)

Kinetic energy: $T = \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} M\dot{x}_2^2 + \frac{1}{2} m\dot{x}_3^2$

Potential energy: $V = \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}k(x_3 - x_2)^2$

$\therefore\ L = T - V$

$= \frac{1}{2} m(\dot{x}_1^2 + \dot{x}_3^2) + \frac{1}{2} M\dot{x}_2^2 - \frac{1}{2}k(x_2 - x_1)^2 - \frac{1}{2}k(x_3 - x_2)^2$

b) Lagrange eq'ns. of motion:

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0$$

1: $k(x_2 - x_1) - \frac{d}{dt}(m\dot{x}_1) = 0$

$$\therefore\ \ddot{x}_1 = \frac{k}{m}(x_2 - x_1) \qquad (1)$$

2: $-k(x_2 - x_1)\ \cancel{\frac{d}{dt}(}\ + k(x_3 - x_2) - \frac{d}{dt}(M\dot{x}_2) = 0$

$$\therefore\ \ddot{x}_2 = \frac{k}{M}(x_1 + x_3 - 2x_2) \qquad (2)$$

3: $-k(x_3 - x_2) - \frac{d}{dt}(m\dot{x}_3) = 0$

$$\therefore\ \ddot{x}_3 = \frac{k}{m}(x_2 - x_3) \qquad (3)$$

c) Assume $x_i = A_i \cos\omega t$, substitute in equations (1) – (3):

$$
\begin{cases}
m\omega^2 A_1 + k(A_2 - A_1) = 0 \\
M\omega^2 A_2 + k(A_1 + A_3 - 2A_2) = 0 \\
m\omega^2 A_3 + k(A_2 - A_3) = 0
\end{cases}
$$

or
$$
\begin{pmatrix} m\omega^2 - k & k & 0 \\ k & M\omega^2 - 2k & k \\ 0 & k & m\omega^2 - k \end{pmatrix}
\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0
$$

These simultaneous linear equations have a non-trivial solution iff.

$$
\begin{vmatrix} m\omega^2 - k & k & 0 \\ k & M\omega^2 - 2k & k \\ 0 & k & m\omega^2 - k \end{vmatrix} = 0 \qquad \boxed{QED}
$$

– the secular equation.

One can solve the secular equation to find the eigenvalues (not required):

$$
(m\omega^2 - k)\left[(M\omega^2 - 2k)(m\omega^2 - k) - k^2\right] - k\left[k(m\omega^2 - k)\right] = 0
$$

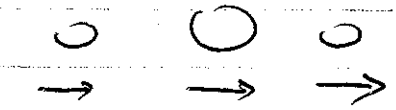
$$
\therefore (m\omega^2 - k)\left[mM\omega^4 - k\omega^2(2m + M) + \cancel{2k^2} \cancel{- k^2} \cancel{- k^2}\right] = 0
$$

$$
\therefore (m\omega^2 - k)\,\omega^2\left[mM\omega^2 - k(2m + M)\right] = 0.
$$

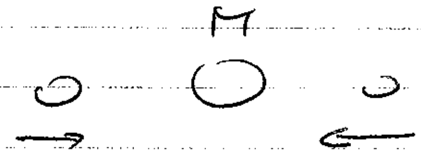
Solutions: $\omega = 0,\ \pm\sqrt{\dfrac{k}{m}},\ \pm\sqrt{\dfrac{k(2m + M)}{mM}}$.

d) These frequencies are the normal frequencies corresponding to the three normal modes of vibration of the system. For the triatomic molecule:

i) $\omega = 0$ corresponds to uniform translation of the whole system, with <u>no</u> relative oscillation:



ii) $\omega_2 = \sqrt{\frac{k}{m}}$ corresponds to the mode:



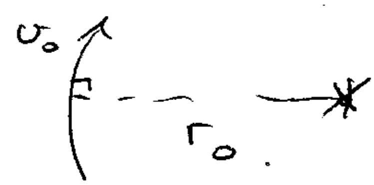
– where M is <u>stationary</u>, & so M does not enter the frequency.

iii) $\omega_3 = \sqrt{\frac{k}{m} + \frac{2k}{M}}$ corresponds to the mode:

– where the centre atom oscillates against the other two.

e) The eigenvectors of the secular equation $\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$ represent the ~~amplitudes~~ relative <u>amplitudes</u> of vibration for each of the normal modes, corresponding to the pictures above.

(4) a) Angular momentum per unit mass at perihelion:



$$l = r^2\dot{\theta} = r_0 v_0$$

$$\therefore \quad r_0 = \frac{m l^2}{k(1+e)} = \frac{m r_0^2 v_0^2}{k(1+e)}$$

$$\therefore e = \frac{m r_0 v_0^2}{k} - 1 \qquad (1) \qquad \boxed{\text{QED}}$$

b) For a circular orbit of radius r_c, Newton's 2nd. law gives

$$F = ma$$

$$\therefore \; -\frac{k}{r_c^2} = -\frac{m v_c^2}{r_c} \qquad (r_c = r_0)$$

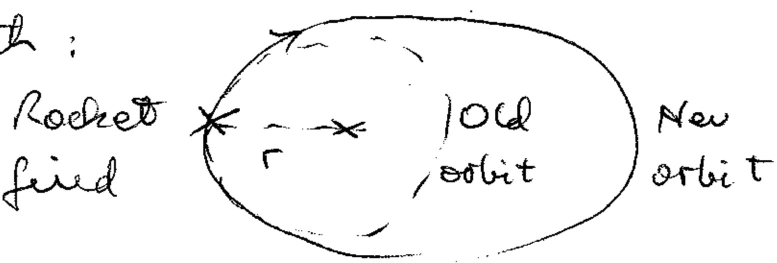
$$\therefore \text{ in (1)}, \quad e = m v_0^2 \left(\frac{1}{m v_c^2} \right) - 1$$

$$= \frac{v_0^2}{v_c^2} - 1 \qquad \boxed{\text{QED}}$$

c) In this case, $v_0 = 1.1 v_c$

$$\therefore \quad e = (1.1)^2 - 1 = 0.21$$

Sketch:



The new orbit is an ellipse, with eccentricity $e = 0.21$, and perihelion $r_0 = r_c$ (the original radius)

d). Aphelion

$$r_1 = r_0\left(\frac{1+e}{1-e}\right) = \frac{1.21}{0.79} r_c = 1.53\, r_c .$$

e) To enter a parabolic orbit would require $e = 1$, i.e.

$$1 = \left(\frac{v_0}{v_c}\right)^2 - 1$$

$$\therefore \frac{v_0}{v_c} = \sqrt{2}$$

i.e. v_0 would need to be 41% greater than v_c.

(5) a) Newton's law: $F = ma$

$$\therefore \; m v \frac{dv}{dx} = F(x)$$

$$\therefore \; m \int v\,dv = \int F(x)\,dx = E - V(x)$$

$$\therefore \; \frac{1}{2} m v^2 = E - V(x)$$

$$\therefore \; v = \sqrt{\frac{2}{m}\left(E - V(x)\right)} \qquad \boxed{\text{QED}}$$

b) $$\underset{\sim}{F} = f(r)\,\underset{\sim}{\hat{r}}.$$

This is a central force problem; hence motion takes place in a plane. Use plane polar co-ordinates
Newton's law:

radial component $\quad m(\ddot{r} - r\dot{\theta}^2) = f(r) \quad (1)$

tangential $\quad$ -- $\quad m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \quad (2)$

Equation (2) implies conservation of angular momentum

$$r^2\dot{\theta} = l \text{ , a constant.}$$

where l is the angular momentum / unit mass.

$\therefore$ in (1) $\quad m\left(\ddot{r} - r\left(\frac{l}{r^2}\right)^2\right) = f(r)$

$$\therefore \; \ddot{r} = \frac{l^2}{r^3} + \frac{f(r)}{m}. \qquad (3)$$

Write $\quad \ddot{r} = m v \frac{dv}{dr} \quad (v = \dot{r})$

following part a)

$$\therefore \int m v \, dv = \int dr \left[f(r) + \frac{m\ell^2}{r^3} \right].$$

$$\text{i.e.} \quad \tfrac{1}{2} m v^2 = E - V_{eff}(r), \qquad (4)$$

where $V_{eff}(r) = -\int dr \left[f(r) + \frac{m\ell^2}{r^3} \right]$

$$\therefore V_{eff}(r) = -\int f(r)\,dr + \frac{m\ell^2}{2r^2} \qquad (5)$$

c) The form of the effective potential is given by (5). The extra term $\frac{m\ell^2}{2r^2}$ is the "centrifugal barrier" term, corresponding to the energy of rotation at angular momentum per unit mass ℓ and radius r.

d). From part b),

$$v = \sqrt{\frac{2}{m}\left(E - V_{eff}(r)\right)}$$

Write $v = \frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = \frac{\ell}{r^2}\frac{dr}{d\theta}$.
(i.e. change variables from t to θ)

$$\therefore \frac{\ell}{r^2}\frac{dr}{d\theta} = \sqrt{\frac{2}{m}\left(E - V_{eff}(r)\right)}$$

$$\therefore \int \frac{\ell}{r^2} \frac{dr}{\sqrt{\frac{2}{m}\left(E - V_{eff}(r)\right)}} = \int d\theta = \theta - \theta_0 . \quad (6)$$

— is the formal solution required.