P.S.

THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

EXAMINATION – JUNE/JULY 2002

PHYS2010 – MECHANICS

Time allowed – 2 hours.

Total number of questions – 5.

Attempt **FOUR** out of **FIVE** questions.

The questions are of equal value.

This paper may be retained by the candidate.

The following materials will be provided by the Enrolments and Assessment Section:
Calculators.

Answers must be written in ink.

Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.

The following equations are supplied as an aid to memory.

Damped Harmonic Motion

If $m\ddot{x} + b\dot{x} + kx = 0$

then $x = Ae^{qt}$

where $q = -\gamma \pm \left(\gamma^2 - \omega_o^2\right)^{1/2}$

$\gamma = b/2m$

$\omega_o^2 = k/m$

$\omega_d^2 = \omega_o^2 - \gamma^2$

Forced Harmonic Motion

If $m\ddot{x} + b\dot{x} + kx = F_o \cos \omega t$

then $x = A\cos(\omega t - \phi)$

where $A = \dfrac{F_o}{\left[m^2\left(\omega_o^2 - \omega^2\right)^2 + b^2\omega^2\right]^{1/2}} = \dfrac{F_o}{m\left[\left(\omega_o^2 - \omega^2\right)^2 + 4\gamma^2\omega^2\right]^{1/2}}$

and $\tan\phi = \dfrac{b\omega}{m\left(\omega_o^2 - \omega^2\right)} = \dfrac{2\gamma\omega}{\left(\omega_o^2 - \omega^2\right)}$

resonance $\omega_r^2 = \omega_o^2 - 2\gamma^2$

$$Q = \frac{\omega_d}{2\gamma}$$

Central Forces

Polar Coords

$\mathbf{r} = (r, \theta)$

$\mathbf{v} = \left(\dot{r},\ r\dot{\theta}\right)$

$\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2, 2\dot{r}\dot{\theta} + r\ddot{\theta}\right)$

$$u = \frac{1}{r} \rightarrow \frac{d^2u}{d\theta^2} + u = -\frac{1}{m\ell^2u^2} f\left(u^{-1}\right)$$

$$r_o = \frac{m\ell^2}{k(1+e)}$$

$$r_1 = r_o\frac{1+e}{1-e}$$

$\ell =$ angular momentum/unit mass

Apsidal angle $\psi = \pi\left(3 + a\dfrac{f'(a)}{f(a)}\right)^{-\frac{1}{2}}$

Stability $f(a)+\frac{a}{3}f'(a)<0$

Inverse Square Law Orbits

$$V=-\frac{k}{r}$$

Gravitation

$$k=GMm$$

$$\dot{\theta}=lu^2$$

$$f(r)=-k/r=-\frac{GMm}{r^2}$$

$$e=\left(1+\frac{2Em\ell^2}{k^2}\right)^{\frac{1}{2}}$$

$$\tau=\frac{2\pi}{\sqrt{GM}}a^{3/2}$$

$$G=6.67\ \text{x}\ 10^{-11}\ N\,m^2\,kg^{-2}$$

$$M_{sun}=2\times10^{30}\ kg$$

$$M_{earth}=6\ x\ 10^{24}\,kg$$

$$R_{earth}=6400\ km$$

Lagrange's Equations

$$L=T-V$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}-\frac{\partial L}{\partial q_j}=0$$

Generalized momenta

$$p_j=\frac{\partial L}{\partial \dot{q}_j}$$

$$H=\sum p_i\,\dot{q}_i-L$$

$$\dot{q}_i=\frac{\partial H}{\partial p_i},\ \dot{p}_i=-\frac{\partial H}{\partial q_i}$$

Question 1

The flow of electrical charge through a series RLC circuit is given by

$$L\ddot{q} + R\dot{q} + \frac{q}{c} = V_0 \exp(i\omega t)$$

where $V = V_0 \exp(i\omega t)$ is the driving voltage from an alternating power source. The circuit elements have values L = 50 mH, R = 20 Ω, C = 20μ F and $V_0 = 50\,V$.

a) Find the natural frequency ω_0 and the damping constant γ of the undriven oscillator. Is the system underdamped, critically damped, or overdamped?

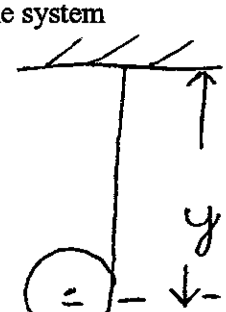
b) Find the resonant frequency of the driven oscillator.

c) What is the Q factor for the system?

d) Sketch the amplitude of the charge oscillations $A(\omega)$ as a function of ω. What is the value at $\omega = 0$?

e) To what value would you adjust the resistance R in order to make the system critically damped?

Question 2



a) A few turns of light string are wrapped around a solid spherical ball of mass m and radius a, with moment of inertia $I = \frac{2}{5} m a^2$. The free end of the string is held fixed while the disk is allowed to fall under gravity, unwinding the string. Find the Langrangian for the system.

b) Use Langrange's equation of motion to show that the acceleration of the ball is given by

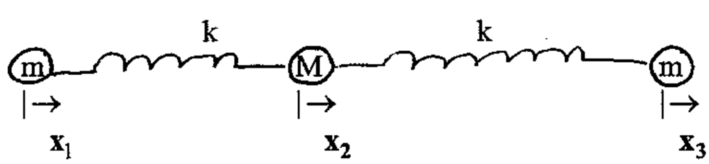
$$\ddot{y} = \frac{5}{7} g$$

c) Find the momentum conjugate to y, and the Hamiltonian for the system

d) Show that the Hamiltonian equation of motion gives the same result as part b).

Question 3

A linear symmetric triatomic molecule consists of two outer particles of mass m and a central one of mass M. The particles are coupled by identical elastic forces, with spring constant k



a) Show that the Lagrangian for this system can be written as

$$L=\frac{1}{2}m(\dot{x}_1^{\,2}+\dot{x}_3^{\,2})+\frac{1}{2}M\dot{x}_2^{\,2}-\frac{1}{2}k(x_2-x_1)^2-\frac{1}{2}k(x_3-x_2)^2$$

b) Use Lagrange's equations to derive the three equations of motion for this system

c) From the three equations of motion, the following secular equation can be obtained:

$$\begin{vmatrix} m\omega^2-k & k & 0 \\ k & M\omega^2-2k & k \\ 0 & k & k \end{vmatrix}=0$$

$m\omega^2-k$

Show how to derive this secular equation.

d) Solving the secular equation gives three angular frequencies:

$$\omega_1=0,\ \omega_2=\left(\frac{k}{m}\right)^{\frac{1}{2}},\ \omega_3=\left(\frac{k}{m}+\frac{2k}{M}\right)^{\frac{1}{2}}$$

Can you deduce the normal modes corresponding to each of these frequencies? Sketch a picture to show the motion in each case.

e) The three frequencies are eigenvalues of the secular matrix. In general terms, what do the corresponding eigenvectors represent?

Question 4

A particle moves in a central, isotropic force field

$$\mathbf{F}=-\frac{k}{r^2}\hat{\mathbf{r}}$$

The solution to the orbit equation for this field is a conic section

$$r=r_0\,\frac{(1+e)}{(1+e\cos\theta)}$$

whose perihelion r_0 is given by

$$r_0=\frac{ml^2}{k(1+e)}$$

where m is the mass of the particle, l is the angular momentum per unit mass, and e is the eccentricity.

a) Find a relation between l and the velocity v_0 of the particle when it is at perihelion, $r=r_0$. Hence show that the eccentricity can be expressed as

$$e=\frac{m r_0 v_0^{\,2}}{k}-1$$

b) Show that the eccentricity can also be expressed

$$e = \left(\frac{v_o}{v_c}\right)^2 - 1$$

where v_c is the velocity corresponding to a circular orbit of radius r_o.

A space craft is in a circular orbit around a planet at radius r_c. It fires a rocket to increase its velocity by 10%.

c) Sketch the new orbit of the space craft. What are the perihelion and eccentricity of the new orbit?

d) Find an expression for the aphelion (maximal distance from the origin) of the new orbit.

e) How much would the space craft in its original circular orbit have to increase its velocity in order to enter into a parabolic trajectory?

Question 5

a) A particle moves in one dimension under the action of a conservative force $F(x)$. Show that the first integral of the equation of motion gives a solution

$$v = \frac{dx}{dt} = \sqrt{\frac{2}{m}(E - V(x)} \qquad (1)$$

b) A central force

$$\mathbf{F} = f(r)\,\hat{\mathbf{r}}$$

is conservative. Show that the central force problem can be reduced to a purely radial problem, with formal solution as in part a), except that the potential $V(r)$ is replaced by an "effective potential".

c) Find an expression for the "effective potential", and explain the physical origin of the extra term.

d) Using the solution of part b) as a starting point, and using the conservation of angular momentum, derive a formal solution for the orbit of the particle relating r and θ.